

---

# pySecDec Documentation

*Release 0.9*

**Sophia Borowka**

**Gudrun Heinrich**

**Stephan Jahn**

**Stephen Jones**

**Matthias Kerner**

**Johannes Schlenk**

**Tom Zirke**

**Mar 02, 2017**



<b>1</b>	<b>Installation</b>	<b>3</b>
1.1	Download the program and install . . . . .	3
1.2	The Geomethod and Normaliz . . . . .	3
1.3	Additional Dependencies for Generated c++ Packages . . . . .	4
<b>2</b>	<b>Getting started</b>	<b>5</b>
2.1	User input . . . . .	5
2.2	Algebraic part and creation of the C++ library . . . . .	6
2.3	Interactive python interface . . . . .	7
<b>3</b>	<b>Overview</b>	<b>9</b>
3.1	pySecDec.algebra . . . . .	9
3.2	Feynman Parametrization of Loop Integrals . . . . .	11
3.3	Sector Decomposition . . . . .	14
3.4	Subtraction . . . . .	17
3.5	Expansion . . . . .	18
<b>4</b>	<b>Reference Guide</b>	<b>21</b>
4.1	Algebra . . . . .	21
4.2	Loop Integral . . . . .	28
4.3	Decomposition . . . . .	34
4.4	Matrix Sort . . . . .	42
4.5	Subtraction . . . . .	43
4.6	Expansion . . . . .	44
4.7	Code Writer . . . . .	45
4.8	Integral Interface . . . . .	49
4.9	Miscellaneous . . . . .	51
<b>5</b>	<b>References</b>	<b>55</b>
<b>6</b>	<b>Indices and tables</b>	<b>57</b>
	<b>Bibliography</b>	<b>59</b>
	<b>Python Module Index</b>	<b>61</b>
	<b>Index</b>	<b>63</b>



*pySecDec* is a toolbox for the calculation of dimensionally regulated parameter integrals using the sector decomposition approach [BH00]; see also [Hei08], [BHJ+15].



## INSTALLATION

### 1.1 Download the program and install

*pySecDec* should run fine with both, *python 2.7* and *python 3* on unix-like systems.

Before you install *pySecDec*, make sure that you have recent versions of *numpy* (<http://www.numpy.org/>) and *sympy* (<http://www.sympy.org/>) installed. In case you want to check for their availability type

```
$ python -c "import numpy"
$ python -c "import sympy" .
```

In case either *numpy* or *sympy* are missing on your machine, it is easiest to install them from your package repository. Alternatively, and in particular if you do not have administrator rights, *pip* (<https://pip.pypa.io/en/stable/>) may be used to perform the installation.

Then download and unpack the tarball from <http://secdec.hepforge.org/>. The tarball contains a distribution of *pySecDec* and the additional dependencies listed *below*. Typing

```
$ make
```

should build all redistributed packages and display two commands to be added to your `.bashrc` or `.profile`.

---

**Note:** Parallel build with `make -j<number-of-cores>` causes trouble on some systems. If `make` finished without a message starting with *Successfully built "pySecDec" and its dependencies*, try again without the `-j` option.

---

### 1.2 The Geomethod and Normaliz

---

**Note:** If you are not urgently interested in using the *geometric decomposition*, you can ignore this section for the beginning. The instructions below are not essential for a *pySecDec* installation. You can still install *normaliz* **after** installing *pySecDec*. All but the *geometric decomposition* routines work without *normaliz*.

---

If you want to use the *geometric decomposition* module, you need the *normaliz [BIR]* command line executable. The *geometric decomposition* module is designed for *normaliz* versions 3.0.0, 3.1.0, and 3.1.1. We recommend to set your `$PATH` such that the *normaliz* executable is found. Alternatively, you can pass the path to the *normaliz* executable directly to the functions that need it.

## 1.3 Additional Dependencies for Generated c++ Packages

The intended main usage of *pySecDec* is to make it write c++ packages using the functions `pySecDec.code_writer.make_package()` and `pySecDec.loop_integral.loop_package()`. In order to build these c++ packages, the following additional non-python-based libraries and programs are required:

- CUBA (<http://www.feynarts.de/cuba/>)
- FORM (<http://www.nikhef.nl/~form/>)
- SecDecUtil (part of *pySecDec*), depends on:
  - catch (<https://github.com/philsquared/Catch>)

The functions `pySecDec.code_writer.make_package()` and `pySecDec.loop_integral.loop_package()` can use the external program *nauty* [BKAP] to find all sector symmetries and therefore reduce the number of sectors:

- NAUTY (<http://pallini.di.uniroma1.it/>)

These packages are redistributed with the *pySecDec* tarball; i.e. you don't have to install any of them yourself.



## GETTING STARTED

After installation, you should have a folder *examples* in your main *pySecDec* directory. It contains various examples, the easiest one being a one-loop box, *box1L.py*. It also contains some two-loop examples: *triangle2L.py*, *box2L.py*, *elliptic\_1L.py*, and examples for parametric functions not related to loop integrals: *Hypergeo5F4.py* calculates Hypergeometric functions, which can have (regulated) poles at both zero and one, *two\_regulators.py* contains an example involving poles in two different regulators. More complex examples are the calculation of the 4-photon amplitude, which shows how to use *pySecDec* as an integral library in a larger context, and the *userdefined\_cpp* example which shows how the user can combine functions to be decomposed with other, user-defined functions.

### 2.1 User input

To explain the input format, let us look at the one-loop box example. The first two lines read

```
import pySecDec as psd
from pySecDec.loop_integral import loop_package
```

They say that the module *pySecDec* should be imported with the alias *psd*, and that the function *loop\_package* from the class *LoopIntegral* is needed.

The following part contains the definition of the loop integral *li*:

```
li = psd.loop_integral.LoopIntegralFromGraph(
# give adjacency list and indicate whether the propagator connecting the numbered_
↳vertices is massive or massless in the first entry of each list item.
internal_lines = [['m', [1,2]], [0, [2,3]], [0, [3,4]], [0, [4,1]]],
# contains the names of the external momenta and the label of the vertex they are_
↳attached to
external_lines = [['p1', 1], ['p2', 2], ['p3', 3], ['p4', 4]],

# define the kinematics and the names for the kinematic invariants
replacement_rules = [
    ('p1*p1', 's1'),
    ('p2*p2', 0),
    ('p3*p3', 0),
    ('p4*p4', 0),
    ('p3*p2', 't/2'),
    ('p1*p2', 's/2-s1/2'),
    ('p1*p4', 't/2-s1/2'),
    ('p2*p4', 's1/2-t/2-s/2'),
    ('p3*p4', 's/2'),
    ('m**2', 'msq')
]
)
```

The symbols for the kinematic invariants and the masses also need to be given as an ordered list. The ordering is important as the numerical values assigned to these list elements at the numerical evaluation stage should have the same order.

```
Mandelstam_symbols = ['s', 't', 's1']
mass_symbols = ['msq']
```

Then the function `loop_package` is called. It will perform the algebraic sector decomposition steps and create a package containing the C++ code for the numerical evaluation. It will create a folder called `box1L` and allows to define parameters controlling the numerical part (for a complete list of possible options see `loop_package`).

```
loop_package(
name = 'box1L',

loop_integral = li,

real_parameters = Mandelstam_symbols + mass_symbols,

# complex_parameters are also possible

# the highest order of the final epsilon expansion
requested_order = 0,

# the optimization level to use in FORM (can be 0, 1, 2, 3)
form_optimization_level = 2,

# the Workspace parameter for FORM
form_work_space = '100M',

# the method to be used for the sector decomposition
# valid values are ``iterative`` and ``geometric``
decomposition_method = 'geometric',

# whether or not to produce code to perform the contour deformation
# if ``True``, it can still be deactivated later in the "config.hpp"
# if ``False``, no code for the contour deformation is generated
contour_deformation = True,
)
```

## 2.2 Algebraic part and creation of the C++ library

Running the python script `box1L.py`

```
$ python box1L.py
```

will create a folder with the name given in `box1L.py` ('`box1L`'), which should contain the following files and subdirectories

```
box1L.hpp  integrate_box1L.cpp  box1L.pdf  codegen  Makefile  Makefile.conf  pylink_
↳ README  src
```

in the folder '`box1L`', typing

```
$ make
```

will create the libraries *libbox1L.a* and *box1L\_pylink.so* which can be linked to an external program calling these integrals. How to do this interactively or via a python script is explained in the next section. In standalone mode, the C++ file *integrate\_box1L.cpp* can be used to produce results for a certain kinematic point. In the latter, kinematic points can be specified by adapting the line

```
const std::vector<box1L::real_t> real_parameters = {9.,-0.1,0.3, 1.};
```

for the desired kinematics. In the above example, the values correspond to  $s=9, t=-0.1, s1=0.3, msq=1$ , i.e. the same ordering is kept as in the lists `Mandelstam_symbols = ['s','t','s1']`, `mass_symbols = ['msq']` in the python input.

The commands

```
$ make integrate_box1L
$ ./integrate_box1L
```

will then evaluate the integral and print the result to the screen.

## 2.3 Interactive python interface

There is also a python interface which allows for an interactive evaluation of the integrals. We will use the 2-loop triangle example to explain how this works:

- first produce the code for the triangle by

```
$ python triangle2L.py
```

- change to the directory `triangle2L` and type

```
$ make
```

- this produces, among other things, the library *triangle2L\_pylink.so*. The latter can be called from within python. In order to do so, `ipython` or `python` can be opened and the following commands can be entered interactively:

```
>>> from __future__ import print_function
>>> from pySecDec.integral_interface import IntegralLibrary
>>> import sympy as sp
>>> # load c++ library
>>> triangle = IntegralLibrary('triangle2L_pylink.so')
```

- now the user can choose an integrator and define the settings for the numerical integration. A list of possible settings is given in `pySecDec.integral_interface`.

```
>>> # choose integrator
>>> triangle.use_Vegas(flags=2,epsrel=1e-3,epsabs=1e-10) # ``flags=2`` means verbose -
↳-> see Cuba manual
```

- the numerical point at which the integral should be evaluated can be given as follows

```
>>> # perform the integration for the numerical point s=0.9, msq=0.1
>>> str_integral_without_prefactor, str_prefactor, str_integral_with_prefactor = _
↳-> triangle(real_parameters=[.9,.1])
```

- the class *triangle* can take more parameters, for example

```
>>> str_integral_with_prefactor = triangle(real_parameters=[.9, .1], number_of_
↳presamples=1e+6, deformation_parameters_maximum = 0.5)
>>> # (defaults: number_of_presamples = 100000, deformation_parameters_maximum = 1)
```

- further options for the contour deformation etc are listed under `pySecDec.integral_interface`
- in addition, the output format can be specified:

```
>>> # convert complex numbers from c++ to sympy notation
>>> str_integral_with_prefactor = str_integral_with_prefactor.replace(',', '+I*')
>>> str_prefactor = str_prefactor.replace(',', '+I*')
>>> str_integral_without_prefactor = str_integral_without_prefactor.replace(',', '+I*
↳')

>>> # convert result to sympy expressions
>>> integral_with_prefactor = sp.sympify(str_integral_with_prefactor.replace('+/-',
↳'*value+error*'))
>>> integral_with_prefactor_err = sp.sympify(str_integral_with_prefactor.replace('+/-
↳', '*value+error*'))
>>> prefactor = sp.sympify(str_prefactor)
>>> integral_without_prefactor = sp.sympify(str_integral_without_prefactor.replace(
↳'+/-', '*value+error*'))
>>> integral_without_prefactor_err = sp.sympify(str_integral_without_prefactor.
↳replace('+/-', '*value+error*'))

>>> # examples how to access individual orders
>>> print('leading pole:', integral_with_prefactor.coeff('eps', -2).coeff('value'),
↳'+/- ('', integral_with_prefactor_err.coeff('eps', -2).coeff('error'), ''))
>>> print('subleading pole:', integral_with_prefactor.coeff('eps', -1).coeff('value
↳'), '+/- ('', integral_with_prefactor_err.coeff('eps', -1).coeff('error'), ''))
>>> print('finite part:', integral_with_prefactor.coeff('eps', 0).coeff('value'), '+/-
↳ ('', integral_with_prefactor_err.coeff('eps', 0).coeff('error'), ''))
```

- This will print the result in a format which is also easy to import into Mathematica. Examples for the above commands are also given in `integrate_triangle.py`.
- How to loop over several kinematic points is shown in the example `multiple_points_example.py`.

## OVERVIEW

*pySecDec* consists of several modules that provide functions and classes for specific purposes. In this overview, we present only the most important aspects of selected modules. These are exactly the modules necessary to set up the algebraic computation of a Feynman loop integral requisite for the numerical evaluation. For detailed instruction of a specific function or class, please be referred to the *reference guide*.

### 3.1 pySecDec.algebra

The *algebra* module implements a very basic computer algebra system. *pySecDec* uses both *sympy* and *numpy*. Although *sympy* in principle provides everything we need, it is way too slow for typical applications. That is because *sympy* is completely written in *python* without making use of any precompiled functions. *pySecDec*'s *algebra* module uses the in general faster *numpy* function wherever possible.

#### 3.1.1 Polynomials

Since sector decomposition is an algorithm that acts on polynomials, we start with the key class *Polynomial*. As the name suggests, the *Polynomial* class is a container for multivariate polynomials, i.e. functions of the form:

$$\sum_i C_i \prod_j x_j^{\alpha_{ij}}$$

A multivariate polynomial is completely determined by its *coefficients*  $C_i$  and the exponents  $\alpha_{ij}$ . The *Polynomial* class stores these in two arrays:

```
>>> from pySecDec.algebra import Polynomial
>>> poly = Polynomial([[1,0], [0,2]], ['A', 'B'])
>>> poly
+ (A)*x0 + (B)*x1**2
>>> poly.explist
array([[1, 0],
       [0, 2]])
>>> poly.coeffs
array([A, B], dtype=object)
```

It is also possible to instantiate the *Polynomial* by its algebraic representation:

```
>>> poly2 = Polynomial.from_expression('A*x0 + B*x1**2', ['x0', 'x1'])
>>> poly2
+ (A)*x0 + (B)*x1**2
>>> poly2.explist
array([[1, 0],
```

```
[0, 2]])
>>> poly2.coeffs
array([A, B], dtype=object)
```

Note that the second argument of `Polynomial.from_expression()` defines the variables  $x_j$ .

Within the `Polynomial` class, basic operations are implemented:

```
>>> poly + 1
+ (1) + (B)*x1**2 + (A)*x0
>>> 2 * poly
+ (2*A)*x0 + (2*B)*x1**2
>>> poly + poly
+ (2*B)*x1**2 + (2*A)*x0
>>> poly * poly
+ (B**2)*x1**4 + (2*A*B)*x0*x1**2 + (A**2)*x0**2
>>> poly ** 2
+ (B**2)*x1**4 + (2*A*B)*x0*x1**2 + (A**2)*x0**2
```

### 3.1.2 General Expressions

In order to perform the `pySecDec.subtraction` and `pySecDec.expansion`, we have to introduce more complex algebraic constructs.

General expressions can be entered in a straightforward way:

```
>>> from pySecDec.algebra import Expression
>>> log_of_x = Expression('log(x)', ['x'])
>>> log_of_x
log( + (1)*x)
```

All expressions in the context of this `algebra` module are based on extending or combining the `Polynomials` introduced *above*. In the example above, `log_of_x` is a `LogOfPolynomial`, which is a derived class from `Polynomial`:

```
>>> type(log_of_x)
<class 'pySecDec.algebra.LogOfPolynomial'>
>>> isinstance(log_of_x, Polynomial)
True
>>> log_of_x.explist
array([[1]])
>>> log_of_x.coeffs
array([1], dtype=object)
```

We have seen an *extension* to the `Polynomial` class, now let us consider a *combination*:

```
>>> more_complex_expression = log_of_x * log_of_x
>>> more_complex_expression
(log( + (1)*x)) * (log( + (1)*x))
```

We just introduced the *Product* of two `LogOfPolynomials`:

```
>>> type(more_complex_expression)
<class 'pySecDec.algebra.Product'>
```

As suggested before, the *Product* combines two `Polynomials`. They are accessible through the *factors*:

```
>>> more_complex_expression.factors[0]
log( + (1)*x)
>>> more_complex_expression.factors[1]
log( + (1)*x)
>>> type(more_complex_expression.factors[0])
<class 'pySecDec.algebra.LogOfPolynomial'>
>>> type(more_complex_expression.factors[1])
<class 'pySecDec.algebra.LogOfPolynomial'>
```

---

**Important:** When working with this *algebra* module, it is important to understand that **everything** is based on the class *Polynomial*.

---

To emphasize the importance of the above statement, consider the following code:

```
>>> expression1 = Expression('x*y', ['x', 'y'])
>>> expression2 = Expression('x*y', ['x'])
>>> type(expression1)
<class 'pySecDec.algebra.Polynomial'>
>>> type(expression2)
<class 'pySecDec.algebra.Polynomial'>
>>> expression1
+ (1)*x*y
>>> expression2
+ (y)*x
```

Although `expression1` and `expression2` are mathematically identical, they are treated differently by the *algebra* module. In `expression1`, both, `x` and `y`, are considered as variables of the *Polynomial*. In contrast, `y` is treated as *coefficient* in `expression2`:

```
>>> expression1.explist
array([[1, 1]])
>>> expression1.coeffs
array([1], dtype=object)
>>> expression2.explist
array([[1]])
>>> expression2.coeffs
array([y], dtype=object)
```

The second argument of the function *Expression* controls how the variables are distributed among the coefficients and the variables in the underlying *Polynomials*. Keep that in mind in order to avoid confusion. One can always check which symbols are considered as variables by asking for the `symbols`:

```
>>> expression1.symbols
[x, y]
>>> expression2.symbols
[x]
```

## 3.2 Feynman Parametrization of Loop Integrals

The primary purpose of *pySecDec* is the numerical calculation of loop integrals as they arise in fixed order calculations in quantum field theories. In the first step of our approach, the loop integral is converted from the momentum representation to the Feynman parameter representation.

The module `pySecDec.loop_integral` implements exactly that conversion. The most basic use is to calculate the first and the second Symanzik polynomial  $U$  and  $F$ , respectively, from the propagators of a loop integral.

### 3.2.1 One Loop Bubble

To calculate  $U$  and  $F$  of the one loop bubble, type the following commands:

```
>>> from pySecDec.loop_integral import LoopIntegralFromPropagators
>>> propagators = ['k**2', '(k - p)**2']
>>> loop_momenta = ['k']
>>> one_loop_bubble = LoopIntegralFromPropagators(propagators, loop_momenta)
>>> one_loop_bubble.U
+ (1)*x0 + (1)*x1
>>> one_loop_bubble.F
+ (-p**2)*x0*x1
```

The example above among other useful features is also stated in the full documentation of `LoopIntegralFromPropagators()` in the reference guide.

### 3.2.2 Two Loop Planar Box with Numerator

Consider the propagators of the two loop planar box:

```
>>> propagators = ['k1**2', '(k1+p2)**2',
...               '(k1-p1)**2', '(k1-k2)**2',
...               '(k2+p2)**2', '(k2-p1)**2',
...               '(k2+p2+p3)**2']
>>> loop_momenta = ['k1', 'k2']
```

We could now instantiate the `LoopIntegral` just like *before*. However, let us consider an additional numerator:

```
>>> numerator = 'k1(mu)*k1(mu) + 2*k1(mu)*p3(mu) + p3(mu)*p3(mu)' # (k1 + p3) ** 2
```

In order to unambiguously define the loop integral, we must state which symbols denote the `Lorentz_indices` (just `mu` in this case here) and the `external_momenta`:

```
>>> external_momenta = ['p1', 'p2', 'p3', 'p4']
>>> Lorentz_indices=['mu']
```

With that, we can Feynman parametrize the two loop box with a numerator:

```
>>> box = LoopIntegralFromPropagators(propagators, loop_momenta, external_momenta,
...                                   numerator=numerator, Lorentz_indices=Lorentz_
↳ indices)
>>> box.U
+ (1)*x3*x6 + (1)*x3*x5 + (1)*x3*x4 + (1)*x2*x6 + (1)*x2*x5 + (1)*x2*x4 + (1)*x2*x3
↳ + (1)*x1*x6 + (1)*x1*x5 + (1)*x1*x4 + (1)*x1*x3 + (1)*x0*x6 + (1)*x0*x5 + (1)*x0*x4
↳ + (1)*x0*x3
>>> box.F
+ (-p1**2 - 2*p1*p2 - 2*p1*p3 - p2**2 - 2*p2*p3 - p3**2)*x3*x5*x6 + (-
↳ p3**2)*x3*x4*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x3*x4*x5 + (-p1**2 - 2*p1*p2 - 2*p1*p3
↳ - p2**2 - 2*p2*p3 - p3**2)*x2*x5*x6 + (-p3**2)*x2*x4*x6 + (-p1**2 - 2*p1*p2 -
↳ p2**2)*x2*x4*x5 + (-p1**2 - 2*p1*p2 - 2*p1*p3 - p2**2 - 2*p2*p3 - p3**2)*x2*x3*x6 +
↳ (-p1**2 - 2*p1*p2 - p2**2)*x2*x3*x4 + (-p1**2 - 2*p1*p2 - 2*p1*p3 - p2**2 - 2*p2*p3
↳ - p3**2)*x1*x5*x6 + (-p3**2)*x1*x4*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x4*x5 + (-
↳ p3**2)*x1*x3*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x3*x5 + (-p1**2 - 2*p1*p2 -
↳ p2**2)*x1*x2*x6 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x2*x5 + (-p1**2 - 2*p1*p2 -
↳ p2**2)*x1*x2*x4 + (-p1**2 - 2*p1*p2 - p2**2)*x1*x2*x3 + (-p1**2 - 2*p1*p2 - 2*p1*p3
↳ - p2**2 - 2*p2*p3 - p3**2)*x0*x5*x6 + (-p3**2)*x0*x4*x6 + (-p1**2 - 2*p1*p2 -
↳ p2**2)*x0*x4*x5 + (-p2**2 - 2*p2*p3 - p3**2)*x0*x3*x6 + (-p1**2 - 2*p1*p2 -
↳ p2**2)*x0*x3*x5 + (-
↳ p2**2)*x0*x3*x4 + (-p1**2)*x0*x2*x6 + (-p1**2)*x0*x2*x5 + (-p1**2)*x0*x2*x4 + (-
↳ p1**2)*x0*x2*x3 + (-p2**2)*x0*x1*x6 + (-p2**2)*x0*x1*x5 + (-p2**2)*x0*x1*x4 + (-
↳ p2**2)*x0*x1*x3
```



```

>>> box.numerator
+ (-2*eps*p3(mu)**2 - 2*p3(mu)**2)*U**2 + (-eps + 2)*x6*F + (-eps + 2)*x5*F + (-eps
↪+ 2)*x4*F + (-eps + 2)*x3*F + (4*eps*p2(mu)*p3(mu) + 4*eps*p3(mu)**2 +
↪4*p2(mu)*p3(mu) + 4*p3(mu)**2)*x3*x6*U + (-4*eps*p1(mu)*p3(mu) -
↪4*p1(mu)*p3(mu))*x3*x5*U + (4*eps*p2(mu)*p3(mu) + 4*p2(mu)*p3(mu))*x3*x4*U + (-
↪2*eps*p2(mu)**2 - 4*eps*p2(mu)*p3(mu) - 2*eps*p3(mu)**2 - 2*p2(mu)**2 -
↪4*p2(mu)*p3(mu) - 2*p3(mu)**2)*x3**2*x6**2 + (4*eps*p1(mu)*p2(mu) +
↪4*eps*p1(mu)*p3(mu) + 4*p1(mu)*p2(mu) + 4*p1(mu)*p3(mu))*x3**2*x5*x6 + (-
↪2*eps*p1(mu)**2 - 2*p1(mu)**2)*x3**2*x5**2 + (-4*eps*p2(mu)**2 -
↪4*eps*p2(mu)*p3(mu) - 4*p2(mu)**2 - 4*p2(mu)*p3(mu))*x3**2*x4*x6 +
↪(4*eps*p1(mu)*p2(mu) + 4*p1(mu)*p2(mu))*x3**2*x4*x5 + (-2*eps*p2(mu)**2 -
↪2*p2(mu)**2)*x3**2*x4**2 + (-4*eps*p1(mu)*p3(mu) - 4*p1(mu)*p3(mu))*x2*x6*U + (-
↪4*eps*p1(mu)*p3(mu) - 4*p1(mu)*p3(mu))*x2*x5*U + (-4*eps*p1(mu)*p3(mu) -
↪4*p1(mu)*p3(mu))*x2*x4*U + (-4*eps*p1(mu)*p3(mu) - 4*p1(mu)*p3(mu))*x2*x3*U +
↪(4*eps*p1(mu)*p2(mu) + 4*eps*p1(mu)*p3(mu) + 4*p1(mu)*p2(mu) +
↪4*p1(mu)*p3(mu))*x2*x3*x6**2 + (-4*eps*p1(mu)**2 + 4*eps*p1(mu)*p2(mu) +
↪4*eps*p1(mu)*p3(mu) - 4*p1(mu)**2 + 4*p1(mu)*p2(mu) + 4*p1(mu)*p3(mu))*x2*x3*x5*x6
↪+ (-4*eps*p1(mu)**2 - 4*p1(mu)**2)*x2*x3*x5**2 + (8*eps*p1(mu)*p2(mu) +
↪4*eps*p1(mu)*p3(mu) + 8*p1(mu)*p2(mu) + 4*p1(mu)*p3(mu))*x2*x3*x4*x6 + (-
↪4*eps*p1(mu)**2 + 4*eps*p1(mu)*p2(mu) - 4*p1(mu)**2 + 4*p1(mu)*p2(mu))*x2*x3*x4*x5
↪+ (4*eps*p1(mu)*p2(mu) + 4*p1(mu)*p2(mu))*x2*x3*x4**2 + (4*eps*p1(mu)*p2(mu) +
↪4*eps*p1(mu)*p3(mu) + 4*p1(mu)*p2(mu) + 4*p1(mu)*p3(mu))*x2*x3**2*x6 + (-
↪4*eps*p1(mu)**2 - 4*p1(mu)**2)*x2*x3**2*x5 + (4*eps*p1(mu)*p2(mu) +
↪4*p1(mu)*p2(mu))*x2*x3**2*x4 + (-2*eps*p1(mu)**2 - 2*p1(mu)**2)*x2**2*x6**2 + (-
↪4*eps*p1(mu)**2 - 4*p1(mu)**2)*x2**2*x5*x6 + (-2*eps*p1(mu)**2 -
↪4*eps*p1(mu)**2)*x2**2*x4*x6 + (-4*eps*p1(mu)**2 - 4*p1(mu)**2)*x2**2*x3*x6 + (-
↪4*eps*p1(mu)**2 - 4*p1(mu)**2)*x2**2*x3*x5 + (-4*eps*p1(mu)**2 -
↪4*p1(mu)**2)*x2**2*x3*x4 + (-2*eps*p1(mu)**2 - 2*p1(mu)**2)*x2**2*x3**2 +
↪(4*eps*p2(mu)*p3(mu) + 4*p2(mu)*p3(mu))*x1*x6*U + (4*eps*p2(mu)*p3(mu) +
↪4*p2(mu)*p3(mu))*x1*x5*U + (4*eps*p2(mu)*p3(mu) + 4*p2(mu)*p3(mu))*x1*x4*U +
↪(4*eps*p2(mu)*p3(mu) + 4*p2(mu)*p3(mu))*x1*x3*U + (-4*eps*p2(mu)**2 -
↪4*eps*p2(mu)*p3(mu) - 4*p2(mu)**2 - 4*p2(mu)*p3(mu))*x1*x3*x6**2 +
↪(4*eps*p1(mu)*p2(mu) - 4*eps*p2(mu)**2 - 4*eps*p2(mu)*p3(mu) + 4*p1(mu)*p2(mu) -
↪4*p2(mu)**2 - 4*p2(mu)*p3(mu))*x1*x3*x5*x6 + (4*eps*p1(mu)*p2(mu) +
↪4*p1(mu)*p2(mu))*x1*x3*x5**2 + (-8*eps*p2(mu)**2 - 4*eps*p2(mu)*p3(mu) -
↪8*p2(mu)**2 - 4*p2(mu)*p3(mu))*x1*x3*x4*x6 + (4*eps*p1(mu)*p2(mu) - 4*eps*p2(mu)**2
↪+ 4*p1(mu)*p2(mu) - 4*p2(mu)**2)*x1*x3*x4*x5 + (-4*eps*p2(mu)**2 -
↪4*eps*p2(mu)*p3(mu) - 4*p2(mu)**2 - 4*p2(mu)*p3(mu))*x1*x3*x4**2 + (-4*eps*p2(mu)**2 -
↪4*eps*p2(mu)*p3(mu) + (4*eps*p1(mu)*p2(mu) + 4*p1(mu)*p2(mu))*x1*x3**2*x5
↪+ (-4*eps*p2(mu)**2 - 4*p2(mu)**2)*x1*x3**2*x4 + (4*eps*p1(mu)*p2(mu) +
↪4*p1(mu)*p2(mu))*x1*x2*x6**2 + (8*eps*p1(mu)*p2(mu) + 8*p1(mu)*p2(mu))*x1*x2*x5*x6
↪+ (4*eps*p1(mu)*p2(mu) + 4*p1(mu)*p2(mu))*x1*x2*x5**2 + (8*eps*p1(mu)*p2(mu) +
↪8*p1(mu)*p2(mu))*x1*x2*x4*x6 + (8*eps*p1(mu)*p2(mu) + 8*p1(mu)*p2(mu))*x1*x2*x4*x5
↪+ (4*eps*p1(mu)*p2(mu) + 4*p1(mu)*p2(mu))*x1*x2*x4**2 + (8*eps*p1(mu)*p2(mu) +
↪8*p1(mu)*p2(mu))*x1*x2*x3*x6 + (8*eps*p1(mu)*p2(mu) + 8*p1(mu)*p2(mu))*x1*x2*x3*x5
↪+ (8*eps*p1(mu)*p2(mu) + 8*p1(mu)*p2(mu))*x1*x2*x3*x4 + (4*eps*p1(mu)*p2(mu) +
↪4*p1(mu)*p2(mu))*x1*x2*x3**2 + (-2*eps*p2(mu)**2 - 2*p2(mu)**2)*x1**2*x6**2 + (-
↪4*eps*p2(mu)**2 - 4*p2(mu)**2)*x1**2*x5*x6 + (-2*eps*p2(mu)**2 -
↪4*eps*p2(mu)**2)*x1**2*x4*x6 + (-4*eps*p2(mu)**2 - 4*p2(mu)**2)*x1**2*x4*x5 + (-
↪2*eps*p2(mu)**2 - 4*p2(mu)**2)*x1**2*x4**2 + (-4*eps*p2(mu)**2 - 4*p2(mu)**2)*x1**2*x3*x6 + (-
↪4*eps*p2(mu)**2 - 4*p2(mu)**2)*x1**2*x3*x5 + (-4*eps*p2(mu)**2 -
↪4*p2(mu)**2)*x1**2*x3*x4 + (-2*eps*p2(mu)**2 - 2*p2(mu)**2)*x1**2*x3**2

```

We can also generate the output in terms of Mandelstam invariants:

```

>>> replacement_rules = [
...     ('p1*p1', 0),
...     ('p2*p2', 0),
...     ('p3*p3', 0),
...     ('p4*p4', 0),
...     ('p1*p2', 's/2'),
...     ('p2*p3', 't/2'),
...     ('p1*p3', '-s/2-t/2')
... ]
>>> box = LoopIntegralFromPropagators(propagators, loop_momenta, external_momenta,
...     numerator=numerator, Lorentz_indices=Lorentz_
↳ indices,
...     replacement_rules=replacement_rules)
>>> box.U
+ (1)*x3*x6 + (1)*x3*x5 + (1)*x3*x4 + (1)*x2*x6 + (1)*x2*x5 + (1)*x2*x4 + (1)*x2*x3
↳ + (1)*x1*x6 + (1)*x1*x5 + (1)*x1*x4 + (1)*x1*x3 + (1)*x0*x6 + (1)*x0*x5 + (1)*x0*x4
↳ + (1)*x0*x3
>>> box.F
+ (-s)*x3*x4*x5 + (-s)*x2*x4*x5 + (-s)*x2*x3*x4 + (-s)*x1*x4*x5 + (-s)*x1*x3*x5 + (-
↳ s)*x1*x2*x6 + (-s)*x1*x2*x5 + (-s)*x1*x2*x4 + (-s)*x1*x2*x3 + (-s)*x0*x4*x5 + (-
↳ t)*x0*x3*x6
>>> box.numerator
+ (-eps + 2)*x6*F + (-eps + 2)*x5*F + (-eps + 2)*x4*F + (-eps + 2)*x3*F + (2*eps*t +
↳ 2*t)*x3*x6*U + (-4*eps*(-s/2 - t/2) + 2*s + 2*t)*x3*x5*U + (2*eps*t + 2*t)*x3*x4*U
↳ + (-2*eps*t - 2*t)*x3**2*x6**2 + (2*eps*s + 4*eps*(-s/2 - t/2) - 2*t)*x3**2*x5*x6 +
↳ (-2*eps*t - 2*t)*x3**2*x4*x6 + (2*eps*s + 2*s)*x3**2*x4*x5 + (-4*eps*(-s/2 - t/2) +
↳ 2*s + 2*t)*x2*x6*U + (-4*eps*(-s/2 - t/2) + 2*s + 2*t)*x2*x5*U + (-4*eps*(-s/2 - t/
↳ 2) + 2*s + 2*t)*x2*x4*U + (-4*eps*(-s/2 - t/2) + 2*s + 2*t)*x2*x3*U + (2*eps*s +
↳ 4*eps*(-s/2 - t/2) - 2*t)*x2*x3*x6**2 + (2*eps*s + 4*eps*(-s/2 - t/2) -
↳ 2*t)*x2*x3*x5*x6 + (4*eps*s + 4*eps*(-s/2 - t/2) + 2*s - 2*t)*x2*x3*x4*x6 +
↳ (2*eps*s + 2*s)*x2*x3*x4*x5 + (2*eps*s + 2*s)*x2*x3*x4**2 + (2*eps*s + 4*eps*(-s/2 -
↳ t/2) - 2*t)*x2*x3**2*x6 + (2*eps*s + 2*s)*x2*x3**2*x4 + (2*eps*t + 2*t)*x1*x6*U +
↳ (2*eps*t + 2*t)*x1*x5*U + (2*eps*t + 2*t)*x1*x4*U + (2*eps*t + 2*t)*x1*x3*U + (-
↳ 2*eps*t - 2*t)*x1*x3*x6**2 + (2*eps*s - 2*eps*t + 2*s - 2*t)*x1*x3*x5*x6 + (2*eps*s
↳ + 2*s)*x1*x3*x5**2 + (-2*eps*t - 2*t)*x1*x3*x4*x6 + (2*eps*s + 2*s)*x1*x3*x4*x5 + (-
↳ 2*eps*t - 2*t)*x1*x3**2*x6 + (2*eps*s + 2*s)*x1*x3**2*x5 + (2*eps*s +
↳ 2*s)*x1*x2*x6**2 + (4*eps*s + 4*s)*x1*x2*x5*x6 + (2*eps*s + 2*s)*x1*x2*x5**2 +
↳ (4*eps*s + 4*s)*x1*x2*x4*x6 + (4*eps*s + 4*s)*x1*x2*x4*x5 + (2*eps*s +
↳ 2*s)*x1*x2*x4**2 + (4*eps*s + 4*s)*x1*x2*x3*x6 + (4*eps*s + 4*s)*x1*x2*x3*x5 +
↳ (4*eps*s + 4*s)*x1*x2*x3*x4 + (2*eps*s + 2*s)*x1*x2*x3**2

```

### 3.3 Sector Decomposition

The sector decomposition algorithm aims to factorize the polynomials  $P_i$  as products of a monomial and a polynomial with nonzero constant term:

$$P_i(\{x_j\}) \mapsto \prod_j x_j^{\alpha_j} (const + p_i(\{x_j\})).$$

Factorizing polynomials in that way by exploiting integral transformations is the first step in an algorithm for solving dimensionally regulated integrals of the form

$$\int_0^1 \prod_{i,j} P_i(\{x_j\})^{\beta_i} dx_j.$$

The iterative sector decomposition splits the integral and remaps the integration domain until all polynomials  $P_i$  in all arising integrals (called *sectors*) have the desired form *const + polynomial*. An introduction to the sector decomposition approach can be found in [Hei08].

To demonstrate the `pySecDec.decomposition` module, we decompose the polynomials

```
>>> p1 = Polynomial.from_expression('x + A*y', ['x', 'y', 'z'])
>>> p2 = Polynomial.from_expression('x + B*y*z', ['x', 'y', 'z'])
```

Let us first focus on the iterative decomposition of `p1`. In the `pySecDec` framework, we first have to pack `p1` into a `Sector`:

```
>>> from pySecDec.decomposition import Sector
>>> initial_sector = Sector([p1])
>>> print(initial_sector)
Sector:
Jacobian= + (1)
cast=[( + (1)) * ( + (1)*x + (A)*y)]
other=[]
```

We can now run the iterative decomposition and take a look at the decomposed sectors:

```
>>> from pySecDec.decomposition.iterative import iterative_decomposition
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (A)*y)]
other=[]

Sector:
Jacobian= + (1)*y
cast=[( + (1)*y) * ( + (1)*x + (A))]
other=[]
```

The decomposition of `p2` needs two iterations and yields three sectors:

```
>>> initial_sector = Sector([p2])
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (B)*y*z)]
other=[]

Sector:
Jacobian= + (1)*x*y
cast=[( + (1)*x*y) * ( + (1) + (B)*z)]
other=[]
```

```
Sector:
Jacobian= + (1)*y*z
cast=[( + (1)*y*z) * ( + (1)*x + (B))]
other=[]
```

Note that we declared  $z$  as a variable for sector  $p1$  even though it does not depend on it. This declaration is necessary if we want to simultaneously decompose  $p1$  and  $p2$ :

```
>>> initial_sector = Sector([p1, p2])
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (A)*y), ( + (1)*x) * ( + (1) + (B)*y*z)]
other=[]

Sector:
Jacobian= + (1)*x*y
cast=[( + (1)*y) * ( + (1)*x + (A)), ( + (1)*x*y) * ( + (1) + (B)*z)]
other=[]

Sector:
Jacobian= + (1)*y*z
cast=[( + (1)*y) * ( + (1)*x*z + (A)), ( + (1)*y*z) * ( + (1)*x + (B))]
other=[]
```

We just fully decomposed  $p1$  and  $p2$ . In some cases, one may want to bring one polynomial, say  $p1$ , into standard form, but not necessarily the other. For that purpose, the `Sector` can take a second argument. In the following code example, we bring  $p1$  into standard form, apply all transformations to  $p2$  as well, but stop before  $p2$  is fully decomposed:

```
>>> initial_sector = Sector([p1], [p2])
>>> decomposed_sectors = iterative_decomposition(initial_sector)
>>> for sector in decomposed_sectors:
...     print(sector)
...     print('\n')
...
Sector:
Jacobian= + (1)*x
cast=[( + (1)*x) * ( + (1) + (A)*y)]
other=[ + (1)*x + (B)*x*y*z]

Sector:
Jacobian= + (1)*y
cast=[( + (1)*y) * ( + (1)*x + (A))]
other=[ + (1)*x*y + (B)*y*z]
```

### 3.4 Subtraction

In the subtraction, we want to perform those integrations that lead to  $\epsilon$  divergencies. The master formula for one integration variables is

$$\int_0^1 x^{(a-b\epsilon)} \mathcal{I}(x, \epsilon) dx = \sum_{p=0}^{|a|-1} \frac{1}{a+p+1-b\epsilon} \frac{\mathcal{I}^{(p)}(0, \epsilon)}{p!} + \int_0^1 x^{(a-b\epsilon)} R(x, \epsilon) dx$$

where  $\mathcal{I}^{(p)}$  is denotes the p-th derivative of  $\mathcal{I}$  with respect to  $x$ . The equation above effectively defines the remainder term  $R$ . All terms on the right hand side of the equation above are constructed to be free of divergencies. For more details and the generalization to multiple variables, we refer the reader to [Hei08]. In the following, we show how to use the implementation in *pySecDec*.

To initialize the subtraction, we first define a factorized expression of the form  $x^{(-1-b_x\epsilon)}y^{(-2-b_y\epsilon)}\mathcal{I}(x, y, \epsilon)$ :

```
>>> from pySecDec.algebra import Expression
>>> symbols = ['x', 'y', 'eps']
>>> x_monomial = Expression('x**(-1 - b_x*eps)', symbols)
>>> y_monomial = Expression('y**(-2 - b_y*eps)', symbols)
>>> cal_I = Expression('cal_I(x, y, eps)', symbols)
```

We must pack the monomials into a *pySecDec.algebra.Product*:

```
>>> from pySecDec.algebra import Product
>>> monomials = Product(x_monomial, y_monomial)
```

Although this seems to be to complete input according to the equation above, we are still missing a structure to store poles in. The function *pySecDec.subtraction.integrate\_pole\_part()* is designed to return an iterable of the same type as the input. That is particularly important since the output of the subtraction of one variable is the input for the subtraction of the next variable. We will see this iteration later. Initially, we do not have poles yet, therefore we define a *one* of the required type:

```
>>> from pySecDec.algebra import Pow
>>> import numpy as np
>>> polynomial_one = Polynomial(np.zeros([1, len(symbols)], dtype=int), np.array([1]),
↳ symbols, copy=False)
>>> pole_part_initializer = Pow(polynomial_one, -polynomial_one)
```

*pole\_part\_initializer* is of type *pySecDec.algebra.Pow* and has *-polynomial\_one* in the exponent. We initialize the *base* with *polynomial\_one*; i.e. a one packed into a polynomial. The function *pySecDec.subtraction.integrate\_pole\_part()* populates the *base* with factors of  $b\epsilon$  when poles arise.

We are now ready to build the *subtraction\_initializer* - the *pySecDec.algebra.Product* to be passed into *pySecDec.subtraction.integrate\_pole\_part()*.

```
>>> from pySecDec.subtraction import integrate_pole_part
>>> subtraction_initializer = Product(monomials, pole_part_initializer, cal_I)
>>> x_subtracted = integrate_pole_part(subtraction_initializer, 0)
```

The second argument of *pySecDec.subtraction.integrate\_pole\_part()* specifies to which variable we want to apply the master formula, here we choose  $x$ . First, remember that the  $x$  monomial is a dimensionally regulated  $x^{-1}$ . Therefore, the sum collapses to only one term and we have two terms in total. Each term corresponds to one entry in the list *x\_subtracted*:

```
>>> len(x_subtracted)
2
```

`x_subtracted` has the same structure as our input. The first factor of each term stores the remaining monomials:

```
>>> x_subtracted[0].factors[0]
(( + (1))**( + (-b_x)*eps + (-1))) * (( + (1)*y))**( + (-b_y)*eps + (-2))
>>> x_subtracted[1].factors[0]
(( + (1)*x))**( + (-b_x)*eps + (-1))) * (( + (1)*y))**( + (-b_y)*eps + (-2))
```

The second factor stores the  $\epsilon$  poles. There is an epsilon pole in the first term, but still none in the second:

```
>>> x_subtracted[0].factors[1]
( + (-b_x)*eps) ** ( + (-1))
>>> x_subtracted[1].factors[1]
( + (1)) ** ( + (-1))
```

The last factor catches everything that is not covered by the first two fields:

```
>>> x_subtracted[0].factors[2]
(cal_I( + (0), + (1)*y, + (1)*eps))
>>> x_subtracted[1].factors[2]
(cal_I( + (1)*x, + (1)*y, + (1)*eps)) + (( + (-1)) * (cal_I( + (0), + (1)*y, +
↪(1)*eps)))
```

We have now performed the subtraction for  $x$ . Because in and output have a similar structure, we can easily perform the subtraction for  $y$  as well:

```
>>> x_and_y_subtracted = []
>>> for s in x_subtracted:
...     x_and_y_subtracted.extend( integrate_pole_part(s,1) )
```

Alternatively, we can directly instruct `pySecDec.subtraction.integrate_pole_part()` to perform both subtractions:

```
>>> alternative_x_and_y_subtracted = integrate_pole_part(subtraction_initializer,0,1)
```

In both cases, the result is a list of the terms appearing on the right hand side of the master equation.

## 3.5 Expansion

The purpose of the `expansion` module is, as the name suggests, to provide routines to perform a series expansion. The module basically implements two routines - the Taylor expansion (`pySecDec.expansion.expand_Taylor()`) and an expansion of polyrational functions supporting singularities in the expansion variable (`pySecDec.expansion.expand_singular()`).

### 3.5.1 Taylor expansion

The function `pySecDec.expansion.expand_Taylor()` implements the ordinary Taylor expansion. It takes an algebraic expression (in the sense of the *algebra module*, the index of the expansion variable and the order to which the expression shall be expanded:

```
>>> from pySecDec.algebra import Expression
>>> from pySecDec.expansion import expand_Taylor
>>> expression = Expression('x**eps', ['eps'])
>>> expand_Taylor(expression, 0, 2).simplify()
+ (1) + (log( + (x)))*eps + ((log( + (x))) * (log( + (x))) * ( + (1/2)))*eps**2
```

It is also possible to expand an expression in multiple variables simultaneously:

```
>>> expression = Expression('x**(eps + alpha)', ['eps', 'alpha'])
>>> expand_Taylor(expression, [0,1], [2,0]).simplify()
+ (1) + (log( + (x)))*eps + ((log( + (x))) * (log( + (x))) * ( + (1/2)))*eps**2
```

The command above instructs `pySecDec.expansion.expand_Taylor()` to expand the expression to the second order in the variable indexed 0 (eps) and to the zeroth order in the variable indexed 1 (alpha).

### 3.5.2 Laurent Expansion

`pySecDec.expansion.expand_singular()` Laurent expands polyrational functions.

Its input is more restrictive than for the *Taylor expansion*. It expects a *Product* where the factors are either *Polynomials* or *ExponentiatedPolynomials* with exponent = -1:

```
>>> from pySecDec.expansion import expand_singular
>>> expression = Expression('1/(eps + alpha)', ['eps', 'alpha']).simplify()
>>> expand_singular(expression, 0, 1)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 241, in _
↳expand_singular
    return _expand_and_flatten(product, indices, orders, _expand_singular_step)
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 209, in _
↳expand_and_flatten
    expansion = recursive_expansion(expression, indices, orders)
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 198, in _
↳recursive_expansion
    expansion = expansion_one_variable(expression, index, order)
  File "/home/pcl340a/sjahn/Projects/pySecDec/pySecDec/expansion.py", line 82, in _
↳expand_singular_step
    raise TypeError("`product` must be a `Product`')
TypeError: `product` must be a `Product`
>>> expression # ``expression`` is indeed a polyrational function.
( + (1)*alpha + (1)*eps)**(-1)
>>> type(expression) # It is just not packed in a ``Product`` as ``expand_singular``
↳expects.
<class 'pySecDec.algebra.ExponentiatedPolynomial'>
>>> from pySecDec.algebra import Product
>>> expression = Product(expression)
>>> expand_singular(expression, 0, 1)
+ (( + (1)) * (( + (1)*alpha)**(-1))) + (( + (-1)) * (( + (1)*alpha**2)**(-1)))*eps
```

Like in the *Taylor expansion*, we can expand simultaneously in multiple parameters. Note, however, that the result of the Laurent expansion depends on the ordering of the expansion variables. The second argument of `pySecDec.expansion.expand_singular()` determines the order of the expansion:

```
>>> expression = Expression('1/(2*eps) * 1/(eps + alpha)', ['eps', 'alpha']).
↳simplify()
>>> eps_first = expand_singular(expression, [0,1], [1,1])
>>> eps_first
+ (( + (1/2)) * (( + (1))**(-1)))*eps**(-1)*alpha**(-1) + (( + (-1/2)) * (( + (1))**(-
↳1))*alpha**(-2) + (( + (1)) * (( + (2))**(-1)))*eps*alpha**(-3)
>>> alpha_first = expand_singular(expression, [1,0], [1,1])
>>> alpha_first
+ (( + (1/2)) * (( + (1))**(-1)))*eps**(-2) + (( + (-1/2)) * (( + (1))**(-1)))*eps**(-
↳3)*alpha
```

The expression printed out by our algebra module are quite messy. In order to obtain nicer output, we can convert these expressions to the slower but more high level *sympy*:

```
>>> import sympy as sp
>>> eps_first = expand_singular(expression, [0,1], [1,1])
>>> alpha_first = expand_singular(expression, [1,0], [1,1])
>>> sp.simplify(eps_first)
1/(2*alpha*eps) - 1/(2*alpha**2) + eps/(2*alpha**3)
>>> sp.simplify(alpha_first)
-alpha/(2*eps**3) + 1/(2*eps**2)
```



## REFERENCE GUIDE

This section describes all public functions and classes in *pySecDec*.

### 4.1 Algebra

Implementation of a simple computer algebra system.

**class** `pySecDec.algebra.ExponentiatedPolynomial` (*explist*, *coeffs*, *exponent=1*, *polysymbols='x'*, *copy=True*)

Like *Polynomial*, but with a global exponent.  $polynomial^{exponent}$

#### Parameters

- **explist** – iterable of iterables; The variable’s powers for each term.
- **coeffs** – iterable; The coefficients of the polynomial.
- **exponent** – object, optional; The global exponent.
- **polysymbols** – iterable or string, optional; The symbols to be used for the polynomial variables when converted to string. If a string is passed, the variables will be consecutively numbered.

**For example:** `explist=[[2,0],[1,1]] coeffs=["A","B"]`

– `polysymbols='x'` (default)  $\leftrightarrow$  “ $A*x^0**2 + B*x^0*x^1$ ”

– `polysymbols=['x','y']`  $\leftrightarrow$  “ $A*x**2 + B*x*y$ ”

- **copy** – bool; Whether or not to copy the *explist*, the *coeffs*, and the *exponent*.

---

**Note:** If `copy` is `False`, it is assumed that the *explist*, the *coeffs* and the *exponent* have the correct type.

---

**copy** ()

Return a copy of a *Polynomial* or a subclass.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*.

**Parameters** *index* – integer; The index of the parameter to derive by.

**simplify** ()

Apply the identity  $\langle \text{something} \rangle^{**0} = 1$  or  $\langle \text{something} \rangle^{**1} = \langle \text{something} \rangle$  or  $1^{**}\langle \text{something} \rangle = 1$  if possible, otherwise call the `simplify` method of the base class. Convert *exponent* to symbol if possible.

pySecDec.algebra.**Expression** (*expression, polysymbols, follow\_functions=False*)

Convert a sympy expression to an expression in terms of this module.

**Parameters**

- **expression** – string or sympy expression; The expression to be converted
- **polysymbols** – iterable of strings or sympy symbols; The symbols to be stored as *expolists* (see *Polynomial*) where possible.
- **follow\_functions** – bool, optional (default = False); If true, return the converted expression and a list of *Function* that occur in the *expression*.

class pySecDec.algebra.**Function** (*symbol, \*arguments, \*\*kwargs*)

Symbolic function that can take care of parameter transformations. It keeps track of all taken derivatives: When *derive()* is called, save the multiindex of the taken derivative.

The derivative multiindices are the keys in the dictionary *self.derivative\_tracks*. The values are lists with two elements: Its first element is the index to derive the derivative indicated by the multiindex in the second element by, in order to obtain the derivative indicated by the key:

```
>>> from pySecDec.algebra import Polynomial, Function
>>> x = Polynomial.from_expression('x', ['x', 'y'])
>>> y = Polynomial.from_expression('y', ['x', 'y'])
>>> poly = x**2*y + y**2
>>> func = Function('f', x, y)
>>> ddfuncd0d1 = func.derive(0).derive(1)
>>> func
Function(f( + (1)*x, + (1)*y), derivative_tracks = {(1, 0): [0, (0, 0)], (1, 1): [1, (1, 0)]})
>>> func.derivative_tracks
{(1, 0): [0, (0, 0)], (1, 1): [1, (1, 0)]}
>>> func.compute_derivatives(poly)
{(1, 0): + (2)*x*y, (1, 1): + (2)*x}
```

**Parameters**

- **symbol** – string; The symbol to be used to represent the *Function*.
- **arguments** – arbitrarily many *\_Expression*; The arguments of the *Function*.
- **copy** – bool; Whether or not to copy the *arguments*.

**compute\_derivatives** (*expression=None*)

Compute all derivatives of *expression* that are mentioned in *self.derivative\_tracks*. The purpose of this function is to avoid computing the same derivatives multiple times.

**Parameters** **expression** – *\_Expression*, optional; The expression to compute the derivatives of. If not provided, the derivatives are shown as in terms of the *function*'s derivatives *dfd<index>*.

**copy()**

Return a copy of a *Function*.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*. The derivative of a function with *symbol* *f* by some *index* is denoted as *dfd<index>*.

**Parameters** **index** – integer; The index of the paramater to derive by.

**replace** (*expression*, *index*, *value*, *remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

#### Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

**simplify** ()

Simplify the arguments.

**class** `pySecDec.algebra.Log` (*arg*, *copy=True*)

The (natural) logarithm to base e (2.718281828459..). Store the expressions `log(arg)`.

#### Parameters

- **arg** – `_Expression`; The argument of the logarithm.
- **copy** – bool; Whether or not to copy the *arg*.

**copy** ()

Return a copy of a *Log*.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*.

**Parameters** **index** – integer; The index of the parameter to derive by.

**replace** (*expression*, *index*, *value*, *remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

#### Parameters

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

**simplify** ()

Apply `log(1) = 0`.

**class** `pySecDec.algebra.LogOfPolynomial` (*expolist*, *coeffs*, *polysymbols='x'*, *copy=True*)

The natural logarithm of a *Polynomial*.

#### Parameters

- **expolist** – iterable of iterables; The variable's powers for each term.
- **coeffs** – iterable; The coefficients of the polynomial.
- **exponent** – object, optional; The global exponent.

- **polysymbols** – iterable or string, optional; The symbols to be used for the polynomial variables when converted to string. If a string is passed, the variables will be consecutively numbered.

**For example: `expolist=[[2,0],[1,1]] coeffs=["A","B"]`**

– `polysymbols='x'` (default) <-> `"A*x0**2 + B*x0*x1"`

– `polysymbols=['x','y']` <-> `"A*x**2 + B*x*y"`

**derive** (*index*)

Generate the derivative by the parameter indexed *index*.

**Parameters** **index** – integer; The index of the paramater to derive by.

**static from\_expression** (*expression, polysymbols*)

Alternative constructor. Construct the *LogOfPolynomial* from an algebraic expression.

**Parameters**

- **expression** – string or sympy expression; The algebraic representation of the polynomial, e.g. `"5*x1**2 + x1*x2"`
- **polysymbols** – iterable of strings or sympy symbols; The symbols to be interpreted as the polynomial variables, e.g. `['x1','x2']`.

**simplify** ()

Apply the identity  $\log(1) = 0$ , otherwise call the simplify method of the base class.

**class** `pySecDec.algebra.Polynomial` (*expolist, coeffs, polysymbols='x', copy=True*)

Container class for polynomials. Store a polynomial as list of lists counting the powers of the variables. For example the polynomial `"x1**2 + x1*x2"` is stored as `[[2,0],[1,1]]`.

Coefficients are stored in a separate list of strings, e.g. `"A*x0**2 + B*x0*x1"` <-> `[[2,0],[1,1]]` and `["A","B"]`.

**Parameters**

- **expolist** – iterable of iterables; The variable's powers for each term.

---

**Hint:** Negative powers are allowed.

---

- **coeffs** – 1d array-like with numerical or sympy-symbolic (see <http://www.sympy.org/>) content, e.g. `[x,1,2]` where `x` is a sympy symbol; The coefficients of the polynomial.
- **polysymbols** – iterable or string, optional; The symbols to be used for the polynomial variables when converted to string. If a string is passed, the variables will be consecutively numbered.

**For example: `expolist=[[2,0],[1,1]] coeffs=["A","B"]`**

– `polysymbols='x'` (default) <-> `"A*x0**2 + B*x0*x1"`

– `polysymbols=['x','y']` <-> `"A*x**2 + B*x*y"`

- **copy** – bool; Whether or not to copy the *expolist* and the *coeffs*.

---

**Note:** If `copy` is `False`, it is assumed that the *expolist* and the *coeffs* have the correct type.

---

**becomes\_zero\_for** (*zero\_params*)

Return True if the polynomial becomes zero if the parameters passed in *zero\_params* are set to zero. Otherwise, return False.

**Parameters** `zero_params` – iterable of integers; The indices of the parameters to be checked.

`copy()`

Return a copy of a *Polynomial* or a subclass.

`derive(index)`

Generate the derivative by the parameter indexed *index*.

**Parameters** `index` – integer; The index of the parameter to derive by.

`static from_expression(expression, polysymbols)`

Alternative constructor. Construct the polynomial from an algebraic expression.

**Parameters**

- **expression** – string or sympy expression; The algebraic representation of the polynomial, e.g. “ $5*x1^{**2} + x1*x2$ ”
- **polysymbols** – iterable of strings or sympy symbols; The symbols to be interpreted as the polynomial variables, e.g. “[‘x1’, ‘x2’]”.

`has_constant_term(indices=None)`

Return True if the polynomial can be written as:

$$const + \dots$$

Otherwise, return False.

**Parameters** `indices` – list of integers or None; The indices of the *polysymbols* to consider. If None (default) all indices are taken into account.

`replace(expression, index, value, remove=False)`

Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

**Parameters**

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

`simplify(deep=True)`

Combine terms that have the same exponents of the variables.

**Parameters** `deep` – bool; If True (default) call the *simplify* method of the coefficients if they are of type `_Expression`.

`class pySecDec.algebra.Pow(base, exponent, copy=True)`

Exponential. Store two expressions A and B to be interpreted as the exponential  $A^{**}B$ .

**Parameters**

- **base** – `_Expression`; The base A of the exponential.
- **exponent** – `_Expression`; The exponent B.
- **copy** – bool; Whether or not to copy *base* and *exponent*.

`copy()`

Return a copy of a *Pow*.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*.

**Parameters** *index* – integer; The index of the parameter to derive by.

**replace** (*expression, index, value, remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

**Parameters**

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

**simplify** ()

Apply the identity  $\langle \text{something} \rangle^{**0} = 1$  or  $\langle \text{something} \rangle^{**1} = \langle \text{something} \rangle$  or  $1^{**}\langle \text{something} \rangle = 1$  if possible. Convert to *ExponentiatedPolynomial* or *Polynomial* if possible.

**class** `pySecDec.algebra.Product` (*\*factors, \*\*kwargs*)

Product of polynomials. Store one or polynomials  $p_i$  to be interpreted as product  $\prod_i p_i$ .

**Parameters**

- **factors** – arbitrarily many instances of *Polynomial*; The factors  $p_i$ .
- **copy** – bool; Whether or not to copy the *factors*.

$p_i$  can be accessed with `self.factors[i]`.

Example:

```
p = Product(p0, p1)
p0 = p.factors[0]
p1 = p.factors[1]
```

**copy** ()

Return a copy of a *Product*.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*. Return an instance of the optimized *ProductRule*.

**Parameters** *index* – integer; The index of the parameter to derive by.

**replace** (*expression, index, value, remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

**Parameters**

- **expression** – `_Expression`; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.

- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

**simplify()**

If one or more of `self.factors` is a *Product*, replace it by its factors. If only one factor is present, return that factor. Remove factors of one and zero.

**class** `pySecDec.algebra.ProductRule` (\*expressions, \*\*kwargs)

Store an expression of the form

$$\sum_i c_i \prod_j \prod_k \left( \frac{d}{dx_k} \right)^{n_{ijk}} f_j(\{x_k\})$$

The main reason for introducing this class is a speedup when calculating derivatives. In particular, this class implements simplifications such that the number of terms grows less than exponentially (scaling of the naive implementation of the product rule) with the number of derivatives.

**Parameters** `expressions` – arbitrarily many expressions; The expressions  $f_j$ .

**copy()**

Return a copy of a *ProductRule*.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*. Note that this class is particularly designed to hold derivatives of a product.

**Parameters** `index` – integer; The index of the parameter to derive by.

**replace** (*index*, *value*, *remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all `expolist` of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the `expolist`.

**Parameters**

- **expression** – *Expression*; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the `parameters` in the *expression*.

**simplify()**

Combine terms that have the same derivatives of the *expressions*.

**to\_sum()**

Convert the *ProductRule* to *Sum*

**class** `pySecDec.algebra.Sum` (\*summands, \*\*kwargs)

Sum of polynomials. Store one or polynomials  $p_i$  to be interpreted as product  $\sum_i p_i$ .

**Parameters**

- **summands** – arbitrarily many instances of *Polynomial*; The summands  $p_i$ .
- **copy** – bool; Whether or not to copy the *summands*.

$p_i$  can be accessed with `self.summands[i]`.

Example:

```
p = Sum(p0, p1)
p0 = p.summands[0]
p1 = p.summands[1]
```

**copy()**

Return a copy of a *Sum*.

**derive** (*index*)

Generate the derivative by the parameter indexed *index*.

**Parameters** *index* – integer; The index of the parameter to derive by.

**replace** (*expression, index, value, remove=False*)

Replace a variable in an expression by a number or a symbol. The entries in all *expolist* of the underlying *Polynomial* are set to zero. The coefficients are modified according to *value* and the powers indicated in the *expolist*.

**Parameters**

- **expression** – *\_Expression*; The expression to replace the variable.
- **index** – integer; The index of the variable to be replaced.
- **value** – number or sympy expression; The value to insert for the chosen variable.
- **remove** – bool; Whether or not to remove the replaced parameter from the *parameters* in the *expression*.

**simplify()**

If one or more of *self.summands* is a *Sum*, replace it by its summands. If only one summand is present, return that summand. Remove zero from sums.

## 4.2 Loop Integral

This module defines routines to Feynman parametrize a loop integral and build a c++ package that numerically integrates over the sector decomposed integrand.

### 4.2.1 Feynman Parametrization

Routines to Feynman parametrize a loop integral.

**class** `pySecDec.loop_integral.LoopIntegral` (*\*args, \*\*kwargs*)

Container class for loop integrals. The main purpose of this class is to convert a loop integral from the momentum representation to the Feynman parameter representation.

It is possible to provide either the graph of the loop integrals as adjacency list, or the propagators.

The Feynman parametrized integral is a product of the following expressions that are accessible as member properties:

- `self.regulator ** self.regulator_power`
- `self.Gamma_factor`
- `self.exponentiated_U`
- `self.exponentiated_F`
- `self.numerator`



- `self.measure`,

where `self` is an instance of either `LoopIntegralFromGraph` or `LoopIntegralFromPropagators`.

When inverse propagators or nonnumerical propagator powers are present (see `powerlist`), some *Feynman parameters* drop out of the integral. The variables to integrate over can be accessed as `self.integration_variables`.

While `self.numerator` describes the numerator polynomial generated by tensor numerators or inverse propagators, `self.measure` contains the monomial associated with the integration measure in the case of propagator powers  $\neq 1$ . The Gamma functions in the denominator belonging to the measure, however, are multiplied to the overall Gamma factor given by `self.Gamma_factor`. The overall sign  $(-1)^{N_\nu}$  is included in `self.numerator`.

**See also:**

- input as graph: `LoopIntegralFromGraph`
- input as list of propagators: `LoopIntegralFromPropagators`

```
class pySecDec.loop_integral.LoopIntegralFromGraph (internal_lines,          external_lines,
                                                    replacement_rules=[],          Feyn-
                                                    man_parameters='x',          regula-
                                                    tor='eps',          regulator_power=0,
                                                    dimensionality='4-2*eps',          pow-
                                                    erlist=[])
```

Construct the Feynman parametrization of a loop integral from the graph using the cut construction method.

Example:

```
>>> from pySecDec.loop_integral import *
>>> internal_lines = [['0', [1, 2]], ['m', [2, 3]], ['m', [3, 1]]]
>>> external_lines = [['p1', 1], ['p2', 2], ['-p12', 3]]
>>> li = LoopIntegralFromGraph(internal_lines, external_lines)
>>> li.exponentiated_U
( + (1)*x0 + (1)*x1 + (1)*x2)**(2*eps - 1)
>>> li.exponentiated_F
( + (m**2)*x2**2 + (2*m**2 - p12**2)*x1*x2 + (m**2)*x1**2 + (m**2 - p1**2)*x0*x2_
↪+ (m**2 - p2**2)*x0*x1)**(-eps - 1)
```

### Parameters

- **internal\_lines** – iterable of internal line specification, consisting of string or sympy expression for mass and a pair of strings or numbers for the vertices, e.g. `[['m', [1, 2]], ['0', [2, 1]]]`.
- **external\_lines** – iterable of external line specification, consisting of string or sympy expression for external momentum and a strings or number for the vertex, e.g. `[['p1', 1], ['p2', 2]]`.
- **replacement\_rules** – iterable of iterables with two strings or sympy expressions, optional; Symbolic replacements to be made for the external momenta, e.g. definition of Mandelstam variables. Example: `[('p1*p2', 's'), ('p1**2', 0)]` where `p1` and `p2` are external momenta. It is also possible to specify vector replacements, for example `[('p4', '(p1+p2+p3)')]`.
- **Feynman\_parameters** – iterable or string, optional; The symbols to be used for the Feynman parameters. If a string is passed, the Feynman parameter variables will be consecutively numbered starting from zero.

- **regulator** – string or sympy symbol, optional; The symbol to be used for the dimensional regulator (typically  $\epsilon$  or  $\epsilon_D$ )

---

**Note:** If you change this symbol, you have to adapt the *dimensionality* accordingly.

---

- **regulator\_power** – integer; The regulator to this power will be multiplied by the numerator.
- **dimensionality** – string or sympy expression, optional; The dimensionality; typically  $4 - 2\epsilon$ , which is the default value.
- **powerlist** – iterable, optional; The powers of the propagators, possibly dependent on the *regulator*. In case of negative powers, the derivative with respect to the corresponding Feynman parameter is calculated as explained in Section 3.2.4 of Ref. [BHJ+15]. If negative powers are combined with a tensor numerator, the derivative acts on the Feynman-parametrized tensor numerator as well, which leads to a consistent result.

```
class pySecDec.loop_integral.LoopIntegralFromPropagators (propagators, loop_momenta,
                                                         external_momenta=[],
                                                         Lorentz_indices=[], numerator=1,
                                                         metric_tensor='g', replacement_rules=[],
                                                         Feynman_parameters='x', regulator='eps',
                                                         regulator_power=0, dimensionality='4-2*eps',
                                                         powerlist=[])
```

Construct the Feynman parametrization of a loop integral from the algebraic momentum representation.

**See also:**

[Hei08], [GKR+11]

Example:

```
>>> from pySecDec.loop_integral import *
>>> propagators = ['k**2', '(k - p)**2']
>>> loop_momenta = ['k']
>>> li = LoopIntegralFromPropagators(propagators, loop_momenta)
>>> li.exponentiated_U
( + (1)*x0 + (1)*x1)**(2*eps - 2)
>>> li.exponentiated_F
( + (-p**2)*x0*x1)**(-eps)
```

The 1st (U) and 2nd (F) Symanzik polynomials and their exponents can also be accessed independently:

```
>>> li.U
+ (1)*x0 + (1)*x1
>>> li.F
+ (-p**2)*x0*x1
>>>
>>> li.exponent_U
2*eps - 2
>>> li.exponent_F
-eps
```

## Parameters

- **propagators** – iterable of strings or sympy expressions; The propagators, e.g. [`'k1**2'`, `'(k1-k2)**2 - m1**2'`].
- **loop\_momenta** – iterable of strings or sympy expressions; The loop momenta, e.g. [`'k1'`, `'k2'`].
- **external\_momenta** – iterable of strings or sympy expressions, optional; The external momenta, e.g. [`'p1'`, `'p2'`]. Specifying the *external\_momenta* is only required when a *numerator* is to be constructed.
- **Lorentz\_indices** – iterable of strings or sympy expressions, optional; Symbols to be used as Lorentz indices in the numerator.

**See also:**

parameter *numerator*

**Parameters numerator** – string or sympy expression, optional; The numerator of the loop integral. Scalar products must be passed in index notation e.g. `"k1(mu)*k2(mu)"`. The numerator should be a sum of products of exclusively: \* numbers \* scalar products (e.g. `"p1(mu)*k1(mu)*p1(nu)*k2(nu)"`) \* *symbols* (e.g. `"m"`)

**Examples:**

- `'p1(mu)*k1(mu)*p1(nu)*k2(nu) + 4*s*eps*k1(mu)*k1(mu)'`
- `'p1(mu)*(k1(mu) + k2(mu))*p1(nu)*k2(nu)'`
- `'p1(mu)*k1(mu)*my_function(eps)'`

---

**Hint:** It is possible to use numbers as indices, for example `'p1(mu)*p2(mu)*k1(nu)*k2(nu) = p1(1)*p2(1)*k1(2)*k2(2)'`.

---



---

**Hint:** The numerator may have uncontracted indices, e.g. `'k1(mu)*k2(nu)'`

---

**Warning:** All Lorentz indices (including the contracted ones) must be explicitly defined using the parameter *Lorentz\_indices*.

**Warning:** It is assumed that the numerator is and all its derivatives by the *regulator* are finite and defined if  $\epsilon = 0$  is inserted explicitly. In particular, if user defined functions (like in the example `p1(mu)*k1(mu)*my_function(eps)`) appear, make sure that `my_function(0)` is finite.

---

**Hint:** In order to mimic a singular user defined function, use the parameter *regulator\_power*. For example, instead of `numerator = gamma(eps)` you could enter `numerator = eps_times_gamma(eps)` in conjunction with `regulator_power = -1`

---

**Warning:** The *numerator* is very flexible in its input. However, that flexibility comes for the price of less error safety. We have no way of checking for all possible mistakes in the input. If your numerator is more advanced than in the examples above, you should proceed with great caution.

### Parameters

- **metric\_tensor** – string or sympy symbol, optional; The symbol to be used for the (Minkowski) metric tensor  $g^{\mu\nu}$ .
- **replacement\_rules** – iterable of iterables with two strings or sympy expressions, optional; Symbolic replacements to be made for the external momenta, e.g. definition of Mandelstam variables. Example: [(‘p1\*p2’, ‘s’), (‘p1\*\*2’, 0)] where p1 and p2 are external momenta. It is also possible to specify vector replacements, for example [(‘p4’, ‘-(p1+p2+p3)’)].
- **Feynman\_parameters** – iterable or string, optional; The symbols to be used for the Feynman parameters. If a string is passed, the Feynman parameter variables will be consecutively numbered starting from zero.
- **regulator** – string or sympy symbol, optional; The symbol to be used for the dimensional regulator (typically  $\epsilon$  or  $\epsilon_D$ )

---

**Note:** If you change this symbol, you have to adapt the *dimensionality* accordingly.

---

- **regulator\_power** – integer; The regulator to this power will be multiplied by the numerator.
- **dimensionality** – string or sympy expression, optional; The dimensionality; typically  $4 - 2\epsilon$ , which is the default value.
- **powerlist** – iterable, optional; The powers of the propagators, possibly dependent on the *regulator*. In case of negative powers, the derivative with respect to the corresponding Feynman parameter is calculated as explained in Section 3.2.4 of Ref. [BHJ+15]. If negative powers are combined with a tensor numerator, the derivative acts on the Feynman-parametrized tensor numerator as well, which leads to a consistent result.

## 4.2.2 Loop Package

This module contains the function that generates a c++ package.

```
pySecDec.loop_integral.loop_package(name, loop_integral, requested_order,
                                   real_parameters=[], complex_parameters=[], con-
                                   tour_deformation=True, additional_prefactor=1,
                                   form_optimization_level=2, form_work_space='500M',
                                   decomposition_method='iterative', nor-
                                   maliz_executable='normaliz', enforce_complex=False,
                                   split=False, ibp_power_goal=-1, use_dreadnaut=True)
```

Decompose, subtract and expand a Feynman parametrized loop integral. Return it as c++ package.

**See also:**

This function is a wrapper around `pySecDec.code_writer.make_package()`.

### Parameters

- **name** – string; The name of the c++ namespace and the output directory.

- **loop\_integral** – `pySecDec.loop_integral.LoopIntegral`; The loop integral to be computed.
- **requested\_orders** – integer; Compute the expansion in the regulator to this order.
- **real\_parameters** – iterable of strings or sympy symbols, optional; Parameters to be interpreted as real numbers, e.g. Mandelstam invariants and masses.
- **parameters** (*complex*) – iterable of strings or sympy symbols, optional; Parameters to be interpreted as complex numbers. To use the complex mass scheme, define the masses as complex parameters.
- **contour\_deformation** – bool, optional; Whether or not to produce code for contour deformation. Default: `True`.
- **additional\_prefactor** – string or sympy expression, optional; An additional factor to be multiplied to the loop integral. It may depend on the regulator, the *real\_parameters*, and the *complex\_parameters*.
- **form\_optimization\_level** – integer out of the interval [0,3], optional; The optimization level to be used in FORM. Default: 2.
- **form\_work\_space** – string, optional; The FORM WorkSpace. Default: '500M'.
- **decomposition\_method** – string, optional; The strategy to decompose the polynomials. The following strategies are available:
  - 'iterative' (default)
  - 'geometric'
  - 'geometric\_ku'

---

**Note:** For 'geometric' and 'geometric\_ku', the third-party program "normaliz" is needed. See [The Geomethod and Normaliz](#).

---

- **normaliz\_executable** – string, optional; The command to run *normaliz*. *normaliz* is only required if *decomposition\_method* is set to 'geometric' or 'geometric\_ku'. Default: 'normaliz'
- **enforce\_complex** – bool, optional; Whether or not the generated integrand functions should have a complex return type even though they might be purely real. The return type of the integrands is automatically complex if *contour\_deformation* is `True` or if there are *complex\_parameters*. In other cases, the calculation can typically be kept purely real. Most commonly, this flag is needed if  $\log(\langle \text{negative real} \rangle)$  occurs in one of the integrand functions. However, *pySecDec* will suggest setting this flag to `True` in that case. Default: `False`
- **split** – bool, optional; Whether or not to split the integration at 1/2 in order to map singularities from 1 to 0. Set this option to `True` if you have singularities when one or more integration variables are one. Default: `False`
- **ibp\_power\_goal** – integer, optional; The *power\_goal* that is forwarded to *integrate\_by\_parts()*.

This option controls how the subtraction terms are generated. Setting it to `-numpy.inf` disables *integrate\_by\_parts()*, while 0 disables *integrate\_pole\_part()*.

**See also:**

To generate the subtraction terms, this function first calls `integrate_by_parts()` for each integration variable with the give `ibp_power_goal`. Then `integrate_pole_part()` is called.

Default: -1

- **use\_dreadnaut** – bool or string, optional; Whether or not to use `squash_symmetry_redundant_sectors_dreadnaut()` to find sector symmetries. If given a string, interpret that string as the command line executable `dreadnaut`. If True, try `$SECDEC_CONTRIB/bin/dreadnaut` and, if the environment variable `$SECDEC_CONTRIB` is not set, `dreadnaut`. Default: True

## 4.2.3 Drawing Feynman Diagrams

Use the following function to draw Feynman diagrams.

```
pySecDec.loop_integral.draw.plot_diagram(internal_lines, external_lines, filename, powerlist=None, neato='neato', extension='pdf', Gstart=0)
```

Draw a Feynman diagram using Graphviz (neato).

Thanks to Viktor Papara <papara@mpp.mpg.de> for his major contributions to this function.

**Warning:** The target is overwritten without prompt if it exists already.

### Parameters

- **internal\_lines** – list; Adjacency list of internal lines, e.g. `[[ 'm', [ 'a', 4 ] ], [ 'm', [ 4, 5 ] ], [ 'm', [ 'a', 5 ] ], [ 0, [ 1, 2 ] ], [ 0, [ 4, 1 ] ], [ 0, [ 2, 5 ] ]]`
- **external\_lines** – list; Adjacency list of external lines, e.g. `[[ 'p1', 1 ], [ 'p2', 2 ], [ 'p3', 'a' ]]`
- **filename** – string; The name of the output file. The generated file gets this name plus the file *extension*.
- **powerlist** – list, optional; The powers of the propagators defined by the *internal\_lines*.
- **neato** – string, default: “neato”; The shell command to call “neato”.
- **extension** – string, default: “pdf”; The file extension. This also defines the output format.
- **Gstart** – nonnegative int; The is value is passed to “neato” with the “-Gstart” option. Try changing this value if the visualization looks bad.

## 4.3 Decomposition

The core of sector decomposition. This module implements the actual decomposition routines.

### 4.3.1 Common

This module collects routines that are used by multiple decomposition modules.

```
class pySecDec.decomposition.Sector(cast, other=[], Jacobian=None)
    Container class for sectors that arise during the sector decomposition.
```

### Parameters

- **cast** – iterable of `algebra.Product` or of `algebra.Polynomial`; The polynomials to be cast to the form  $\langle \text{monomial} \rangle * (\text{const} + \dots)$
- **other** – iterable of `algebra.Polynomial`, optional; All variable transformations are applied to these polynomials but it is not attempted to achieve the form  $\langle \text{monomial} \rangle * (\text{const} + \dots)$
- **Jacobian** – `algebra.Polynomial` with one term, optional; The Jacobian determinant of this sector. If not provided, the according unit monomial ( $1*x^0*0*x^1^0\dots$ ) is assumed.

`pySecDec.decomposition.squash_symmetry_redundant_sectors_sort` (*sectors*,  
*sort\_function*)

Reduce a list of sectors by squashing duplicates with equal integral.

If two sectors only differ by a permutation of the polysymbols (to be interpreted as integration variables over some interval), then the two sectors integrate to the same value. Thus we can drop one of them and count the other twice. The multiple counting of a sector is accounted for by increasing the coefficient of the Jacobian by one.

Equivalence up to permutation is established by applying the `sort_function` to each sector, this brings them into a canonical form. Sectors with identical canonical forms differ only by a permutation.

Note: whether all symmetries are found depends on the choice of `sort_function`. Neither `pySecDec.matrix_sort.iterative_sort()` nor `pySecDec.matrix_sort.Pak_sort()` find all symmetries.

See also: `squash_symmetry_redundant_sectors_dreadnaut()`

Example:

```
>>> from pySecDec.algebra import Polynomial
>>> from pySecDec.decomposition import Sector
>>> from pySecDec.decomposition import squash_symmetry_redundant_sectors_sort
>>> from pySecDec.matrix_sort import Pak_sort
>>>
>>> poly = Polynomial([(0,1), (1,0)], ['a', 'b'])
>>> swap = Polynomial([(1,0), (0,1)], ['a', 'b'])
>>> Jacobian_poly = Polynomial([(1,0)], [3]) # three
>>> Jacobian_swap = Polynomial([(0,1)], [5]) # five
>>> sectors = (
...     Sector([poly], Jacobian=Jacobian_poly),
...     Sector([swap], Jacobian=Jacobian_swap)
... )
>>>
>>> reduced_sectors = squash_symmetry_redundant_sectors_sort(sectors,
...     Pak_sort)
>>> len(reduced_sectors) # symmetry x0 <--> x1
1
>>> # The Jacobians are added together to account
>>> # for the double counting of the sector.
>>> reduced_sectors[0].Jacobian
+ (8)*x0
```

### Parameters

- **sectors** – iterable of `Sector`; the sectors to be reduced.

- **sort\_function** – `pySecDec.matrix_sort.iterative_sort()` or `pySecDec.matrix_sort.Pak_sort()`; The function to be used for finding a canonical form of the sectors.

```
pySecDec.decomposition.squash_symmetry_redundant_sectors_dreadnaut (sectors,
                                                                    dread-
                                                                    naut='dreadnaut',
                                                                    workdir='dreadnaut_tmp',
                                                                    keep_workdir=False)
```

Reduce a list of sectors by squashing duplicates with equal integral.

Each *Sector* is converted to a *Polynomial* which is represented as a graph following the example of [BKAP] (v2.6 Figure 7, Isotopy of matrices).

We first multiply each polynomial in the sector by a unique tag then sum the polynomials of the sector, this converts a sector to a polynomial. Next, we convert the *explist* of the resulting polynomial to a graph where each unique exponent in the *explist* is considered to be a different symbol. Each unique coefficient in the polynomial's *coeffs* is assigned a vertex and connected to the row vertex of any term it multiplies. The external program *dreadnaut* is then used to bring the graph into a canonical form and provide a hash. Sectors with equivalent hashes may be identical, their canonical graphs are compared and if they are identical the sectors are combined.

---

**Note:** This function calls the command line executable of *dreadnaut* [BKAP]. It has been tested with *dreadnaut* version nauty26r7.

---

See also: `squash_symmetry_redundant_sectors_sort()`

#### Parameters

- **sectors** – iterable of *Sector*; the sectors to be reduced.
- **dreadnaut** – string; The shell command to run *dreadnaut*.
- **workdir** – string; The directory for the communication with *dreadnaut*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

---

**Note:** The communication with *dreadnaut* is done via files.

---

- **keep\_workdir** – bool; Whether or not to delete the *workdir* after execution.

### 4.3.2 Iterative

The iterative sector decomposition routines.

**exception** `pySecDec.decomposition.iterative.EndOfDecomposition`

This exception is raised if the function `iteration_step()` is called although the sector is already in standard form.

`pySecDec.decomposition.iterative.find_singular_set (sector, indices=None)`

Function within the iterative sector decomposition procedure which heuristically chooses an optimal decomposition set. The strategy was introduced in arXiv:hep-ph/0004013 [BH00] and is described in 4.2.2 of arXiv:1410.7939 [Bor14]. Return a list of indices.

#### Parameters



- **sector** – *Sector*; The sector to be decomposed.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.iteration_step` (*sector*, *indices=None*)

Run a single step of the iterative sector decomposition as described in chapter 3.2 (part II) of arXiv:0803.4177v2 [Hei08]. Return an iterator of *Sector* - the arising subsectors.

#### Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.iterative_decomposition` (*sector*, *indices=None*)

Run the iterative sector decomposition as described in chapter 3.2 (part II) of arXiv:0803.4177v2 [Hei08]. Return an iterator of *Sector* - the arising subsectors.

#### Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.primary_decomposition` (*sector*, *indices=None*)

Perform the primary decomposition as described in chapter 3.2 (part I) of arXiv:0803.4177v2 [Hei08]. Return a list of *Sector* - the primary sectors. For  $N$  Feynman parameters, there are  $N$  primary sectors where the  $i$ -th Feynman parameter is set to  $1$  in sector  $i$ .

See also:

`primary_decomposition_polynomial()`

#### Parameters

- **sector** – *Sector*; The container holding the polynomials (typically  $U$  and  $F$ ) to eliminate the Dirac delta from.
- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.primary_decomposition_polynomial` (*polynomial*,  
*indices=None*)

Perform the primary decomposition on a single polynomial.

See also:

`primary_decomposition()`

#### Parameters

- **polynomial** – *algebra.Polynomial*; The polynomial to eliminate the Dirac delta from.

- **indices** – iterable of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

`pySecDec.decomposition.iterative.remap_parameters` (*singular\_parameters*, *Jacobian*, *\*polynomials*)

Remap the Feynman parameters according to eq. (16) of arXiv:0803.4177v2 [Hei08]. The parameter whose index comes first in *singular\_parameters* is kept fix.

The remapping is done in place; i.e. the *polynomials* are **NOT** copied.

#### Parameters

- **singular\_parameters** – list of integers; The indices  $\alpha_r$  such that at least one of *polynomials* becomes zero if all  $t_{\alpha_r} \rightarrow 0$ .
- **Jacobian** – *Polynomial*; The Jacobian determinant is multiplied to this polynomial.
- **polynomials** – arbitrarily many instances of *algebra.Polynomial* where all of these have an equal number of variables; The polynomials of Feynman parameters to be remapped. These are typically *F* and *U*.

Example:

```
remap_parameters([1,2], Jacobian, F, U)
```

### 4.3.3 Geometric

The geometric sector decomposition routines.

`pySecDec.decomposition.geometric.Cheng_Wu` (*sector*, *index=-1*)

Replace one Feynman parameter by one. This means integrating out the Dirac delta according to the Cheng-Wu theorem.

#### Parameters

- **sector** – *Sector*; The container holding the polynomials (typically *U* and *F*) to eliminate the Dirac delta from.
- **index** – integer, optional; The index of the Feynman parameter to eliminate. Default: -1 (the last Feynman parameter)

**class** `pySecDec.decomposition.geometric.Polytope` (*vertices=None*, *facets=None*)

Representation of a polytope defined by either its vertices or its facets. Call *complete\_representation()* to translate from one to the other representation.

#### Parameters

- **vertices** – two dimensional array; The polytope in vertex representation. Each row is interpreted as one vertex.
- **facets** – two dimensional array; The polytope in facet representation. Each row represents one facet *F*. A row in *facets* is interpreted as one normal vector  $n_F$  with additionally the constant  $a_F$  in the last column. The points  $v$  of the polytope obey

$$\bigcap_F (\langle n_F, v \rangle + a_F) \geq 0$$

**complete\_representation** (*normaliz*='normaliz', *workdir*='normaliz\_tmp',  
*keep\_workdir*=False)

Transform the vertex representation of a polytope to the facet representation or the other way round. Remove surplus entries in `self.facets` or `self.vertices`.

---

**Note:** This function calls the command line executable of *normaliz* [BIR]. It has been tested with *normaliz* versions 3.0.0, 3.1.0, and 3.1.1.

---

#### Parameters

- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

---

**Note:** The communication with *normaliz* is done via files.

---

- **keep\_workdir** – bool; Whether or not to delete the *workdir* after execution.

**vertex\_incidence\_lists** ()

Return for each vertex the list of facets it lies in (as dictionary). The keys of the output dictionary are the vertices while the values are the indices of the facets in `self.facets`.

`pySecDec.decomposition.geometric.convex_hull` (\**polynomials*)

Calculate the convex hull of the Minkowski sum of all polynomials in the input. The algorithm sets all coefficients to one first and then only keeps terms of the polynomial product that have coefficient 1. Return the list of these entries in the expolist of the product of all input polynomials.

**Parameters polynomials** – arbitrarily many instances of *Polynomial* where all of these have an equal number of variables; The polynomials to calculate the convex hull for.

`pySecDec.decomposition.geometric.generate_fan` (\**polynomials*)

Calculate the fan of the polynomials in the input. The rays of a cone are given by the exponent vectors after factoring out a monomial together with the standard basis vectors. Each choice of factored out monomials gives a different cone. Only full ( $N$ -) dimensional cones in  $R_{\geq 0}^N$  need to be considered.

**Parameters polynomials** – arbitrarily many instances of *Polynomial* where all of these have an equal number of variables; The polynomials to calculate the fan for.

`pySecDec.decomposition.geometric.geometric_decomposition` (*sector*, *indices*=None,  
*normaliz*='normaliz',  
*workdir*='normaliz\_tmp')

Run the sector decomposition using the geomethod as described in [BHJ+15].

---

**Note:** This function calls the command line executable of *normaliz* [BIR]. It has been tested with *normaliz* versions 3.0.0, 3.1.0, and 3.1.1.

---

#### Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – list of integers or None; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.

- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

---

**Note:** The communication with *normaliz* is done via files.

---

```
pySecDec.decomposition.geometric.geometric_decomposition_ku(
    sector,
    indices=None,
    normaliz='normaliz',
    workdir='normaliz_tmp')
```

Run the sector decomposition using the original geometric decomposition strategy by Kaneko and Ueda as described in [KU10].

---

**Note:** This function calls the command line executable of *normaliz* [BIR]. It has been tested with *normaliz* versions 3.0.0, 3.1.0, and 3.1.1.

---

### Parameters

- **sector** – *Sector*; The sector to be decomposed.
- **indices** – list of integers or `None`; The indices of the parameters to be considered as integration variables. By default (`indices=None`), all parameters are considered as integration variables.
- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

---

**Note:** The communication with *normaliz* is done via files.

---

```
pySecDec.decomposition.geometric.transform_variables(
    polynomial,
    transformation,
    polysymbols='y')
```

Transform the parameters  $x_i$  of a `pySecDec.algebra.Polynomial`,

$$x_i \rightarrow \prod_j x_j^{T_{ij}}$$

, where  $T_{ij}$  is the transformation matrix.

### Parameters

- **polynomial** – `pySecDec.algebra.Polynomial`; The polynomial to transform the variables in.
- **transformation** – two dimensional array; The transformation matrix  $T_{ij}$ .
- **polysymbols** – string or iterable of strings; The symbols for the new variables. This argument is passed to the default constructor of `pySecDec.algebra.Polynomial`. Refer to the documentation of `pySecDec.algebra.Polynomial` for further details.

```
pySecDec.decomposition.geometric.triangulate(cone, normaliz='normaliz',
                                             workdir='normaliz_tmp',
                                             keep_workdir=False,
                                             switch_representation=False)
```

Split a cone into simplicial cones; i.e. cones defined by exactly  $D$  rays where  $D$  is the dimensionality.

---

**Note:** This function calls the command line executable of *normaliz* [BIR]. It has been tested with *normaliz* versions 3.0.0, 3.1.0, and 3.1.1.

---

### Parameters

- **cone** – two dimensional array; The defining rays of the cone.
- **normaliz** – string; The shell command to run *normaliz*.
- **workdir** – string; The directory for the communication with *normaliz*. A directory with the specified name will be created in the current working directory. If the specified directory name already exists, an `OSError` is raised.

---

**Note:** The communication with *normaliz* is done via files.

---

- **keep\_workdir** – bool; Whether or not to delete the *workdir* after execution.
- **switch\_representation** – bool; Whether or not to switch between facet and vertex/ray representation.

## 4.3.4 Splitting

Routines to split the integration between 0 and 1. This maps singularities from 1 to 0.

```
pySecDec.decomposition.splitting.find_singular_sets_at_one(polynomial)
```

Find all possible sets of parameters such that the *polynomial*'s constant term vanishes if these parameters are set to one.

Example:

```
>>> from pySecDec.algebra import Polynomial
>>> from pySecDec.decomposition.splitting import find_singular_sets_at_one
>>> polysymbols = ['x0', 'x1']
>>> poly = Polynomial.from_expression('1 - 10*x0 - x1', polysymbols)
>>> find_singular_sets_at_one(poly)
[(1,)]
```

**Parameters** *polynomial* – *Polynomial*; The polynomial to search in.

```
pySecDec.decomposition.splitting.remap_one_to_zero(polynomial, *indices)
```

Apply the transformation  $x \rightarrow 1 - x$  to *polynomial* for the parameters of the given *indices*.

### Parameters

- **polynomial** – *Polynomial*; The polynomial to apply the transformation to.
- **indices** – arbitrarily many int; The indices of the *polynomial.polysymbols* to apply the transformation to.

Example:

```
>>> from pySecDec.algebra import Polynomial
>>> from pySecDec.decomposition.splitting import remap_one_to_zero
>>> polysymbols = ['x0']
>>> polynomial = Polynomial.from_expression('x0', polysymbols)
>>> remap_one_to_zero(polynomial, 0)
+ (1) + (-1)*x0
```

`pySecDec.decomposition.splitting.split` (*sector*, *seed*, \**indices*)

Split the integration interval  $[0, 1]$  for the parameters given by *indices*. The splitting point is fixed using *numpy*'s random number generator.

Return an iterator of *Sector* - the arising subsectors.

**Parameters** *sector* – *Sector*; The sector to be split.

**:param seed**; integer; The seed for the random number generator that is used to fix the splitting point.

**Parameters** *indices* – arbitrarily many integers; The indices of the variables to be split.

`pySecDec.decomposition.splitting.split_singular` (*sector*, *seed*, *indices*=[])

Split the integration interval  $[0, 1]$  for the parameters that can lead to singularities at one for the polynomials in *sector*.cast.

Return an iterator of *Sector* - the arising subsectors.

**Parameters**

- **sector** – *Sector*; The sector to be split.
- **seed** – integer; The seed for the random number generator that is used to fix the splitting point.
- **indices** – iterables of integers; The indices of the variables to be split if required. An empty iterator means that all variables may potentially be split.

## 4.4 Matrix Sort

Algorithms to sort a matrix when column and row permutations are allowed.

`pySecDec.matrix_sort.Pak_sort` (*matrix*)

Inplace modify the *matrix* to some ordering, when permutations of rows and columns (excluding the first) are allowed. The implementation of this function is described in chapter 2 of [Pak11].

---

**Note:** This function may result in different orderings depending on the initial ordering.

---

**See also:**

`iterative_sort()`

**Parameters** *matrix* – 2D array-like; The matrix to be canonicalized.

`pySecDec.matrix_sort.iterative_sort` (*matrix*)

Inplace modify the *matrix* to some ordering, when permutations of rows and columns (excluding the first) are allowed.

---

**Note:** This function may result in different orderings depending on the initial ordering.

---

**See also:**

`Pak_sort()`

**Parameters** `matrix` – 2D array-like; The matrix to be canonicalized.

## 4.5 Subtraction

Routines to isolate the divergencies in an  $\epsilon$  expansion.

`pySecDec.subtraction.integrate_by_parts` (*polyprod*, *power\_goal*, \**indices*)

Repeatedly apply integration by parts,

$$\int_0^1 dt_j t_j^{(a-b\epsilon_1-c\epsilon_2+\dots)} \mathcal{I}(t_j, \{t_{i \neq j}\}, \epsilon_1, \epsilon_2, \dots) = \frac{1}{a+1-b\epsilon_1-c\epsilon_2-\dots} \left( \mathcal{I}(1, \{t_{i \neq j}\}, \epsilon_1, \epsilon_2, \dots) - \int_0^1 dt_j t_j^{(a+1-b\epsilon_1-c\epsilon_2+\dots)} \mathcal{I}'(t_j, \{t_{i \neq j}\}, \epsilon_1, \epsilon_2, \dots) \right)$$

, where  $\mathcal{I}'$  denotes the derivative of  $\mathcal{I}$  with respect to  $t_j$ . The iteration stops, when  $a \geq \text{power\_goal}$ .

**See also:**

This function provides an alternative to `integrate_pole_part()`.

### Parameters

- **polyprod** – `algebra.Product` of the form `<product of <monomial>**(<a_j + ...> * <regulator poles of cal_I> * <cal_I>`; The input product as described above. The `<product of <monomial>**(<a_j + ...>` should be a `pySecDec.algebra.Product` of `<monomial>**(<a_j + ...>` as described below. The `<monomial>**(<a_j + ...>` should be an `pySecDec.algebra.ExponentiatedPolynomial` with exponent being a `Polynomial` of the regulators  $\epsilon_1, \epsilon_2, \dots$ . Although no dependence on the Feynman parameters is expected in the exponent, the polynomial variables should be the Feynman parameters and the regulators. The constant term of the exponent should be numerical. The polynomial variables of `monomial` and the other factors (interpreted as  $\mathcal{I}$ ) are interpreted as the Feynman parameters and the epsilon regulators. Make sure that the last factor (`<cal_I>`) is defined and finite for  $\epsilon = 0$ . All poles for  $\epsilon \rightarrow 0$  should be made explicit by putting them into `<regulator poles of cal_I>` as `pySecDec.algebra.Pow` with exponent = -1 and the base of type `pySecDec.algebra.Polynomial`.
- **indices** – arbitrarily many integers; The index/indices of the parameter(s) to partially integrate.  $j$  in the formulae above.

**Power\_goal** number, e.g. float, integer, ...; The stopping criterion for the iteration.

Return the pole part and the numerically integrable remainder as a list. Each returned list element has the same structure as the input `polyprod`.

`pySecDec.subtraction.integrate_pole_part` (*polyprod*, \**indices*)

Transform an integral of the form

$$\int_0^1 dt_j t_j^{(a-b\epsilon_1-c\epsilon_2+\dots)} \mathcal{I}(t_j, \{t_{i \neq j}\}, \epsilon_1, \epsilon_2, \dots)$$

into the form

$$\sum_{p=0}^{|a|-1} \frac{1}{a+p+1-b\epsilon_1-c\epsilon_2-\dots} \frac{\mathcal{I}^{(p)}(0, \{t_{i \neq j}\}, \epsilon_1, \epsilon_2, \dots)}{p!} + \int_0^1 dt_j t_j^{(a-b\epsilon_1-c\epsilon_2+\dots)} R(t_j, \{t_{i \neq j}\}, \epsilon_1, \epsilon_2, \dots)$$

, where  $\mathcal{I}^{(p)}$  denotes the  $p$ -th derivative of  $\mathcal{I}$  with respect to  $t_j$ . The equations above are to be understood schematically.

**See also:**

This function implements the transformation from equation (19) to (21) as described in arXiv:0803.4177v2 [Hei08].

### Parameters

- **polyprod** – `algebra.Product` of the form `<product of <monomial>** (a_j + ...)> * <regulator poles of cal_I> * <cal_I>`; The input product as described above. The `<product of <monomial>** (a_j + ...)>` should be a `pySecDec.algebra.Product` of `<monomial>** (a_j + ...)` as described below. The `<monomial>** (a_j + ...)` should be an `pySecDec.algebra.ExponentiatedPolynomial` with exponent being a `Polynomial` of the regulators  $\epsilon_1, \epsilon_2, \dots$ . Although no dependence on the Feynman parameters is expected in the exponent, the polynomial variables should be the Feynman parameters and the regulators. The constant term of the exponent should be numerical. The polynomial variables of `monomial` and the other factors (interpreted as  $\mathcal{I}$ ) are interpreted as the Feynman parameters and the epsilon regulators. Make sure that the last factor (`<cal_I>`) is defined and finite for  $\epsilon = 0$ . All poles for  $\epsilon \rightarrow 0$  should be made explicit by putting them into `<regulator poles of cal_I>` as `pySecDec.algebra.Pow` with exponent = -1 and the base of type `pySecDec.algebra.Polynomial`.
- **indices** – arbitrarily many integers; The index/indices of the parameter(s) to partially integrate.  $j$  in the formulae above.

Return the pole part and the numerically integrable remainder as a list. That is the sum and the integrand of equation (21) in arXiv:0803.4177v2 [Hei08]. Each returned list element has the same structure as the input `polyprod`.

`pySecDec.subtraction.pole_structure` (`monomial_product`, `*indices`)

Return a list of the unregulated exponents of the parameters specified by `indices` in `monomial_product`.

### Parameters

- **monomial\_product** – `pySecDec.algebra.ExponentiatedPolynomial` with exponent being a `Polynomial`; The monomials of the subtraction to extract the pole structure from.
- **indices** – arbitrarily many integers; The index/indices of the parameter(s) to partially investigate.

## 4.6 Expansion

Routines to series expand singular and nonsingular expressions.

`pySecDec.expansion.expand_Taylor` (`expression`, `indices`, `orders`)

Series/Taylor expand a nonsingular `expression` around zero.

Return a `algebra.Polynomial` - the series expansion.



**Parameters**

- **expression** – an expression composed of the types defined in the module *algebra*; The expression to be series expanded.
- **indices** – integer or iterable of integers; The indices of the parameters to expand. The ordering of the indices defines the ordering of the expansion.
- **order** – integer or iterable of integers; The order to which the expansion is to be calculated.

`pySecDec.expansion.expand_singular` (*product, indices, orders*)  
Series expand a potentially singular expression of the form

$$\frac{a_N \epsilon_0 + b_N \epsilon_1 + \dots}{a_D \epsilon_0 + b_D \epsilon_1 + \dots}$$

Return a *algebra.Polynomial* - the series expansion.

**See also:**

To expand more general expressions use *expand\_sympy()*.

**Parameters**

- **product** – *algebra.Product* with factors of the form `<polynomial>` and `<polynomial> ** -1`; The expression to be series expanded.
- **indices** – integer or iterable of integers; The indices of the parameters to expand. The ordering of the indices defines the ordering of the expansion.
- **order** – integer or iterable of integers; The order to which the expansion is to be calculated.

`pySecDec.expansion.expand_sympy` (*expression, variables, orders*)  
Expand a sympy expression in the *variables* to given *orders*. Return the expansion as nested *pySecDec.algebra.Polynomial*.

**See also:**

This function is a generalization of *expand\_singular()*.

**Parameters**

- **expression** – string or sympy expression; The expression to be expanded
- **variables** – iterable of strings or sympy symbols; The variables to expand the *expression* in.
- **orders** – iterable of integers; The orders to expand to.

## 4.7 Code Writer

This module collects routines to create a c++ library.

### 4.7.1 Make Package

This is the main function of *pySecDec*.

```
pySecDec.code_writer.make_package(name, integration_variables, regulators, requested_orders,
                                  polynomials_to_decompose, polynomial_names=[],
                                  other_polynomials=[], prefactor=1, remainder_expression=1,
                                  functions=[], real_parameters=[], complex_parameters=[],
                                  form_optimization_level=2, form_work_space='500M',
                                  form_insertion_depth=5, contour_deformation_polynomial=None,
                                  positive_polynomials=[], decomposition_method='iterative_no_primary',
                                  normaliz_executable='normaliz', enforce_complex=False,
                                  split=False, ibp_power_goal=-1, use_dreadnaut=True)
```

Decompose, subtract and expand an expression. Return it as c++ package.

**See also:**

In order to decompose a loop integral, use the function `pySecDec.loop_integral.loop_package()`.

### Parameters

- **name** – string; The name of the c++ namespace and the output directory.
- **integration\_variables** – iterable of strings or sympy symbols; The variables that are to be integrated from 0 to 1.
- **regulators** – iterable of strings or sympy symbols; The UV/IR regulators of the integral.
- **requested\_orders** – iterable of integers; Compute the expansion in the regulators to these orders.
- **polynomials\_to\_decompose** – iterable of strings or sympy expressions or `pySecDec.algebra.ExponentiatedPolynomial` or `pySecDec.algebra.Polynomial`; The polynomials to be decomposed.
- **polynomial\_names** – iterable of strings; Assign symbols for the *polynomials\_to\_decompose*. These can be referenced in the *other\_polynomials*; see *other\_polynomials* for details.
- **other\_polynomials** – iterable of strings or sympy expressions or `pySecDec.algebra.ExponentiatedPolynomial` or `pySecDec.algebra.Polynomial`; Additional polynomials where no decomposition is attempted. The symbols defined in *polynomial\_names* can be used to reference the *polynomials\_to\_decompose*. This is particularly useful when computing loop integrals where the “numerator” can depend on the first and second Symanzik polynomials.

Example (1-loop bubble with numerator):

```
>>> polynomials_to_decompose = ["(x0 + x1)**(2*eps - 4)",
...                             "(-p**2*x0*x1)**(-eps)"]
>>> polynomial_names = ["U", "F"]
>>> other_polynomials = ["(eps - 1)*s*U**2
...                       + (eps - 2)*F
...                       - (eps - 1)*2*s*x0*U
...                       + (eps - 1)*s*x0**2"]
```

**See also:**

`pySecDec.loop_integral`

Note that the *polynomial\_names* refer to the *polynomials\_to\_decompose* **without** their exponents.

- **prefactor** – string or sympy expression, optional; A factor that does not depend on the integration variables.
- **remainder\_expression** – string or sympy expression or `pySecDec.algebra._Expression`, optional; An additional factor.  
Dummy function must be provided with all arguments, e.g. `remainder_expression='exp(eps)*f(x0,x1)'`. In addition, all dummy function must be listed in *functions*.
- **functions** – iterable of strings or sympy symbols, optional; Function symbols occurring in *remainder\_expression*, e.g. `['f']`.

---

**Note:** The power function *pow* and the logarithm *log* are already defined by default. The *log* uses the nonstandard continuation from a negative imaginary part on the negative real axis (e.g.  $\log(-1) = -i\pi$ ).

---

- **real\_parameters** – iterable of strings or sympy symbols, optional; Symbols to be interpreted as real variables.
- **complex\_parameters** – iterable of strings or sympy symbols, optional; Symbols to be interpreted as complex variables.
- **form\_optimization\_level** – integer out of the interval [0,3], optional; The optimization level to be used in FORM. Default: 2.
- **form\_work\_space** – string, optional; The FORM WorkSpace. Default: '500M'.
- **form\_insertion\_depth** – nonnegative integer, optional; How deep FORM should try to resolve nested function calls. Default: 5.
- **contour\_deformation\_polynomial** – string or sympy symbol, optional; The name of the polynomial in *polynomial\_names* that is to be continued to the complex plane according to a  $-i\delta$  prescription. For loop integrals, this is the second Symanzik polynomial  $F$ . If not provided, no code for contour deformation is created.
- **positive\_polynomials** – iterable of strings or sympy symbols, optional; The names of the polynomials in *polynomial\_names* that should always have a positive real part. For loop integrals, this applies to the first Symanzik polynomial  $U$ . If not provided, no polynomial is checked for positiveness. If *contour\_deformation\_polynomial* is `None`, this parameter is ignored.
- **decomposition\_method** – string, optional; The strategy to decompose the polynomials. The following strategies are available:
  - 'iterative\_no\_primary' (default)
  - 'geometric\_no\_primary'
  - 'iterative'
  - 'geometric'
  - 'geometric\_ku'

'iterative', 'geometric', and 'geometric\_ku' are only valid for loop integrals. An end user should always use 'iterative\_no\_primary' or 'geometric\_no\_primary' here. In order to compute loop integrals, please use the function `pySecDec.loop_integral.loop_package()`.

- **normaliz\_executable** – string, optional; The command to run *normaliz*. *normaliz* is only required if *decomposition\_method* starts with ‘geometric’. Default: ‘normaliz’
- **enforce\_complex** – bool, optional; Whether or not the generated integrand functions should have a complex return type even though they might be purely real. The return type of the integrands is automatically complex if *contour\_deformation* is `True` or if there are *complex\_parameters*. In other cases, the calculation can typically be kept purely real. Most commonly, this flag is needed if  $\log(\langle \text{negative real} \rangle)$  occurs in one of the integrand functions. However, *pySecDec* will suggest setting this flag to `True` in that case. Default: `False`
- **split** – bool or integer, optional; Whether or not to split the integration domain in order to map singularities from 1 to 0. Set this option to `True` if you have singularities when one or more integration variables are one. If an integer is passed, that integer is used as seed to generate the splitting point. Default: `False`
- **ibp\_power\_goal** – integer, optional; The *power\_goal* that is forwarded to *integrate\_by\_parts()*.

This option controls how the subtraction terms are generated. Setting it to `-numpy.inf` disables *integrate\_by\_parts()*, while 0 disables *integrate\_pole\_part()*.

**See also:**

To generate the subtraction terms, this function first calls *integrate\_by\_parts()* for each integration variable with the give *ibp\_power\_goal*. Then *integrate\_pole\_part()* is called.

Default: -1

- **use\_dreadnaut** – bool or string, optional; Whether or not to use *squash\_symmetry\_redundant\_sectors\_dreadnaut()* to find sector symmetries. If given a string, interpret that string as the command line executable *dreadnaut*. If `True`, try `$SECDEC_CONTRIB/bin/dreadnaut` and, if the environment variable `$SECDEC_CONTRIB` is not set, *dreadnaut*. Default: `True`

## 4.7.2 Template Parser

Functions to generate c++ sources from template files.

`pySecDec.code_writer.template_parser.parse_template_file(src, dest, replacements={})`

Copy a file from *src* to *dest* replacing `%(...)` instructions in the standard python way.

**Warning:** If the file specified in *dest* exists, it is overwritten without prompt.

**See also:**

*parse\_template\_tree()*

**Parameters**

- **src** – str; The path to the template file.
- **dest** – str; The path to the destination file.
- **replacements** – dict; The replacements to be performed. The standard python replacement rules apply:

```
>>> '%(var)s = %(value)i' % dict(
...     var = 'my_variable',
...     value = 5)
'my_variable = 5'
```

`pySecDec.code_writer.template_parser.parse_template_tree(src, dest, replacements_in_files={}, filesystem_replacements={}, tem_replacements={})`

Copy a directory tree from *src* to *dest* using `parse_template_file()` for each file and replacing the filenames according to `filesystem_replacements`.

See also:

`parse_template_file()`

### Parameters

- **src** – str; The path to the template directory.
- **dest** – str; The path to the destination directory.
- **replacements\_in\_files** – dict; The replacements to be performed in the files. The standard python replacement rules apply:

```
>>> '%(var)s = %(value)i' % dict(
...     var = 'my_variable',
...     value = 5)
'my_variable = 5'
```

- **filesystem\_replacements** – dict; Renaming rules for the destination files. and directories. If a file or directory name in the source tree *src* matches a key in this dictionary, it is renamed to the corresponding value. If the value is `None`, the corresponding file is ignored.

## 4.8 Integral Interface

An interface to libraries generated by `pySecDec.code_writer.make_package()` or `pySecDec.loop_integral.loop_package()`.

**class** `pySecDec.integral_interface.CPPIntegrator`

Abstract base class for integrators to be used with an `IntegralLibrary`. This class holds a pointer to the c++ integrator and defines the destructor.

**class** `pySecDec.integral_interface.Cuhre(integral_library, epsrel=0.01, epsabs=1e-07, flags=0, mineval=0, maxeval=1000000, key=0, real_complex_together=False)`

Wrapper for the Cuhre integrator defined in the cuba library.

**Parameters** `integral_library` – `IntegralLibrary`; The integral to be computed with this integrator.

The other options are defined in the cuba manual.

```
class pySecDec.integral_interface.Divonne(integral_library, epsrel=0.01, epsabs=1e-07,
                                           flags=0, seed=0, mineval=0, maxeval=1000000,
                                           key1=2000, key2=1, key3=1, maxpass=4, border=0.0,
                                           maxchisq=1.0, mindeviation=0.15,
                                           real_complex_together=False)
```

Wrapper for the Divonne integrator defined in the cuba library.

**Parameters** `integral_library` – *IntegralLibrary*; The integral to be computed with this integrator.

The other options are defined in the cuba manual.

```
class pySecDec.integral_interface.IntegralLibrary(shared_object_path)
```

Interface to a c++ library produced by `make_package()` or `loop_package()`.

**Parameters** `shared_object_path` – str; The path to the file “<name>\_pylink.so” that can be built by the command

```
$ make pylink
```

in the root directory of the c++ library.

Instances of this class can be called with the following arguments:

#### Parameters

- **real\_parameters** – iterable of float; The real\_parameters of the library.
- **complex\_parameters** – iterable of complex; The complex parameters of the library.
- **together** – bool; Whether to integrate the sum of all sectors or to integrate the sectors separately. Default: True.
- **number\_of\_presamples** – unsigned int, optional; The number of samples used for the comtour optimization. This option is ignored if the integral library was created without deformation. Default: 100000.
- **deformation\_parameters\_maximum** – float, optional; The maximal value the deformation parameters  $\lambda_i$  can obtain. If `number_of_presamples=0`, all  $\lambda_i$  are set to this value. This option is ignored if the integral library was created without deformation. Default: 1.0.
- **deformation\_parameters\_minimum** – float, optional; This option is ignored if the integral library was created without deformation. Default:  $1e-5$ .
- **deformation\_parameters\_decrease\_factor** – float, optional; If the sign check with the optimized  $\lambda_i$  fails, all  $\lambda_i$  are multiplied by this value until the sign check passes. This option is ignored if the integral library was created without deformation. Default: 0.9.

The call operator returns three strings: \* The integral without its prefactor \* The prefactor \* The integral multiplied by the prefactor

The integrator can be configured by calling the member methods `use_Vegas()`, `use_Suave()`, `use_Divonne()`, and `use_Cuhre()`. The available options are listed in the documentation of *Vegas*, *Suave*, *Divonne*, and *Cuhre*, respectively. If not specified otherwise, *Vegas* is used with its default arguments. For details about the options, refer to the cuba manual.

Further information about the library is stored in the member variable `info` of type `dict`.

```
class pySecDec.integral_interface.Suave(integral_library, epsrel=0.01, epsabs=1e-07,
                                         flags=0, seed=0, mineval=0, maxeval=1000000,
                                         nnew=1000, nmin=10, flatness=25.0,
                                         real_complex_together=False)
```

Wrapper for the Suave integrator defined in the cuba library.

**Parameters** `integral_library` – *IntegralLibrary*; The integral to be computed with this integrator.

The other options are defined in the cuba manual.

```
class pySecDec.integral_interface.Vegas (integral_library, epsrel=0.01, epsabs=1e-07,
                                         flags=0, seed=0, mineval=0, maxeval=1000000,
                                         nstart=1000, nincrease=500, nbatch=1000,
                                         real_complex_together=False)
```

Wrapper for the Vegas integrator defined in the cuba library.

**Parameters** `integral_library` – *IntegralLibrary*; The integral to be computed with this integrator.

The other options are defined in the cuba manual.

## 4.9 Miscellaneous

Collection of general-purpose helper functions.

`pySecDec.misc.adjugate` (*M*)

Calculate the adjugate of a matrix.

**Parameters** *M* – a square-matrix-like array;

`pySecDec.misc.all_pairs` (*iterable*)

Return all possible pairs of a given set. `all_pairs([1, 2, 3, 4]) --> [(1, 2), (3, 4)]`  
`[(1, 3), (2, 4)] [(1, 4), (2, 3)]`

**Parameters** *iterable* – iterable; The set to be split into all possible pairs.

`pySecDec.misc.argsort_2D_array` (*array*)

Sort a 2D array according to its row entries. The idea is to bring identical rows together.

**See also:**

If your array is not two dimensional use `argsort_ND_array()`.

**Example:**

input		sorted
1 2 3		1 2 3
2 3 4		1 2 3
1 2 3		2 3 4

Return the indices like numpy's `argsort()` would.

**Parameters** *array* – 2D array; The array to be argsorted.

`pySecDec.misc.argsort_ND_array` (*array*)

Like `argsort_2D_array()`, this function groups identical entries in an array with any dimensionality greater than (or equal to) two together.

Return the indices like numpy's `argsort()` would.

**See also:**

`argsort_2D_array()`

**Parameters** *array* – ND array,  $N \geq 2$ ; The array to be argsorted.

`pySecDec.misc.assert_degree_at_most_max_degree` (*expression*, *variables*, *max\_degree*, *error\_message*)

Assert that *expression* is a polynomial of degree less or equal *max\_degree* in the *variables*.

`pySecDec.misc.cached_property` (*method*)

Like the builtin *property* to be used as decorator but the method is only called once per instance.

Example:

```
class C(object):
    'Sum up the numbers from one to `N`.'
    def __init__(self, N):
        self.N = N
    @cached_property
    def sum(self):
        result = 0
        for i in range(1, self.N + 1):
            result += i
        return result
```

`pySecDec.misc.det` (*M*)

Calculate the determinant of a matrix.

**Parameters** *M* – a square-matrix-like array;

`pySecDec.misc.doc` (*docstring*)

Decorator that replaces a function's docstring with *docstring*.

Example:

```
@doc('documentation of `some_funcion`')
def some_function(*args, **kwargs):
    pass
```

`pySecDec.misc.lowest_order` (*expression*, *variable*)

Find the lowest order of *expression*'s series expansion in *variable*.

Example:

```
>>> from pySecDec.misc import lowest_order
>>> lowest_order('exp(eps)', 'eps')
0
>>> lowest_order('gamma(eps)', 'eps')
-1
```

#### Parameters

- **expression** – string or sympy expression; The expression to compute the lowest expansion order of.
- **variable** – string or sympy expression; The variable in which to expand.

`pySecDec.misc.missing` (*full*, *part*)

Return the elements in *full* that are not contained in *part*. Raise *ValueError* if an element is in *part* but not in *full*. `missing([1, 2, 3], [1])` --> `[2, 3]` `missing([1, 2, 3, 1], [1, 2])` --> `[3, 1]` `missing([1, 2, 3], [1, 'a'])` --> `ValueError`

#### Parameters

- **full** – iterable; The set of elements to complete *part* with.



- **part** – iterable; The set to be completed to a superset of *full*.

`pySecDec.misc.powerset` (*iterable*, *min\_length=0*, *stride=1*)

Return an iterator over the powerset of a given set. `powerset([1,2,3])` --> `() (1,) (2,) (3,) (1,2) (1,3) (2,3) (1,2,3)`

#### Parameters

- **iterable** – iterable; The set to generate the powerset for.
- **min\_length** – integer, optional; Only generate sets with minimal given length. Default: 0.
- **stride** – integer; Only generate sets that have a multiple of *stride* elements.  
`powerset([1,2,3],stride=2)` --> `() (1,2) (1,3) (2,3)`

`pySecDec.misc.rangecomb` (*low*, *high*)

Return an iterator over the occuring orders in a multivariate series expansion between *low* and *high*.

#### Parameters

- **low** – vector-like array; The lowest orders.
- **high** – vector-like array; The highest orders.

Example:

```
>>> from pySecDec.misc import rangecomb
>>> all_orders = rangecomb([-1,-2], [0,0])
>>> list(all_orders)
[(-1, -2), (-1, -1), (-1, 0), (0, -2), (0, -1), (0, 0)]
```

`pySecDec.misc.sympify_symbols` (*iterable*, *error\_message*, *allow\_number=False*)

*sympify* each item in *iterable* and assert that it is a *symbol*.



**REFERENCES**



## INDICES AND TABLES

- genindex
- modindex
- search



- [BH00] T. Binoth and G. Heinrich *An automatized algorithm to compute infrared divergent multiloop integrals*, Nucl. Phys. B 585 (2000) 741,  
doi:10.1016/S0550-3213(00)00429-6,  
arXiv:hep-ph/0004013
- [BHJ+15] S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke *SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop*, 2015, Comput.Phys.Comm.196  
doi:10.1016/j.cpc.2015.05.022,  
arXiv:1502.06595
- [BIR] W. Bruns and B. Ichim and T. Römer and R. Sieg and C. Söger *Normaliz. Algorithms for rational cones and affine monoids*,  
available at <https://www.normaliz.uni-osnabrueck.de>
- [BKAP] B. D. McKay and A. Piperno \* Practical graph isomorphism, II\*, 2014, Journal of Symbolic Computation, 60, 94-112,  
doi:10.1016/j.jsc.2013.09.003
- [Bor14] S. Borowka *Evaluation of multi-loop multi-scale integrals and phenomenological two-loop applications*, 2014, PhD Thesis - Technische Universität München  
mediaTUM:1220360,  
arXiv:1410.7939
- [GKR+11] J. Gluza, K. Kajda, T. Riemann, V. Yundin *Numerical Evaluation of Tensor Feynman Integrals in Euclidean Kinematics*, 2011, Eur.Phys.J.C71,  
doi:10.1140/epjc/s10052-010-1516-y,  
arXiv:1010.1667
- [Hei08] G. Heinrich *Sector Decomposition*, 2008, Int.J.Mod.Phys.A23,  
doi:10.1142/S0217751X08040263,  
arXiv:0803.4177
- [KU10] T. Kaneko and T. Ueda *A Geometric method of sector decomposition*, 2010, Comput.Phys.Comm.181  
doi:10.1016/j.cpc.2010.04.001,  
arXiv:0908.2897
- [Pak11] A. Pak *The toolbox of modern multi-loop calculations: novel analytic and semi-analytic techniques*, 2012, J. Phys.: Conf. Ser. 368 012049,  
doi:10.1088/1742-6596/368/1/012049,  
arXiv:1111.0868





**a**

pySecDec.algebra, 21

**c**

pySecDec.code\_writer, 45

pySecDec.code\_writer.template\_parser,  
48

**d**

pySecDec.decomposition, 34

pySecDec.decomposition.geometric, 38

pySecDec.decomposition.iterative, 36

pySecDec.decomposition.splitting, 41

**e**

pySecDec.expansion, 44

**i**

pySecDec.integral\_interface, 49

**l**

pySecDec.loop\_integral, 28

**m**

pySecDec.matrix\_sort, 42

pySecDec.misc, 51

**s**

pySecDec.subtraction, 43



**A**

adjugate() (in module pySecDec.misc), 51  
 all\_pairs() (in module pySecDec.misc), 51  
 argsort\_2D\_array() (in module pySecDec.misc), 51  
 argsort\_ND\_array() (in module pySecDec.misc), 51  
 assert\_degree\_at\_most\_max\_degree() (in module py-  
 SecDec.misc), 51

**B**

becomes\_zero\_for() (pySecDec.algebra.Polynomial  
 method), 24

**C**

cached\_property() (in module pySecDec.misc), 52  
 Cheng\_Wu() (in module py-  
 SecDec.decomposition.geometric), 38  
 complete\_representation() (py-  
 SecDec.decomposition.geometric.Polytope  
 method), 38  
 compute\_derivatives() (pySecDec.algebra.Function  
 method), 22  
 convex\_hull() (in module py-  
 SecDec.decomposition.geometric), 39  
 copy() (pySecDec.algebra.ExponentiatedPolynomial  
 method), 21  
 copy() (pySecDec.algebra.Function method), 22  
 copy() (pySecDec.algebra.Log method), 23  
 copy() (pySecDec.algebra.Polynomial method), 25  
 copy() (pySecDec.algebra.Pow method), 25  
 copy() (pySecDec.algebra.Product method), 26  
 copy() (pySecDec.algebra.ProductRule method), 27  
 copy() (pySecDec.algebra.Sum method), 28  
 CPPIntegrator (class in pySecDec.integral\_interface), 49  
 Cuhre (class in pySecDec.integral\_interface), 49

**D**

derive() (pySecDec.algebra.ExponentiatedPolynomial  
 method), 21  
 derive() (pySecDec.algebra.Function method), 22  
 derive() (pySecDec.algebra.Log method), 23  
 derive() (pySecDec.algebra.LogOfPolynomial method),  
 24

derive() (pySecDec.algebra.Polynomial method), 25  
 derive() (pySecDec.algebra.Pow method), 26  
 derive() (pySecDec.algebra.Product method), 26  
 derive() (pySecDec.algebra.ProductRule method), 27  
 derive() (pySecDec.algebra.Sum method), 28  
 det() (in module pySecDec.misc), 52  
 Divonne (class in pySecDec.integral\_interface), 49  
 doc() (in module pySecDec.misc), 52

**E**

EndOfDecomposition, 36  
 expand\_singular() (in module pySecDec.expansion), 45  
 expand\_sympy() (in module pySecDec.expansion), 45  
 expand\_Taylor() (in module pySecDec.expansion), 44  
 ExponentiatedPolynomial (class in pySecDec.algebra),  
 21  
 Expression() (in module pySecDec.algebra), 21

**F**

find\_singular\_set() (in module py-  
 SecDec.decomposition.iterative), 36  
 find\_singular\_sets\_at\_one() (in module py-  
 SecDec.decomposition.splitting), 41  
 from\_expression() (pySecDec.algebra.LogOfPolynomial  
 static method), 24  
 from\_expression() (pySecDec.algebra.Polynomial static  
 method), 25  
 Function (class in pySecDec.algebra), 22

**G**

generate\_fan() (in module py-  
 SecDec.decomposition.geometric), 39  
 geometric\_decomposition() (in module py-  
 SecDec.decomposition.geometric), 39  
 geometric\_decomposition\_ku() (in module py-  
 SecDec.decomposition.geometric), 40

**H**

has\_constant\_term() (pySecDec.algebra.Polynomial  
 method), 25

**I**

IntegralLibrary (class in pySecDec.integral\_interface), 50

integrate\_by\_parts() (in module pySecDec.subtraction), 43  
integrate\_pole\_part() (in module pySecDec.subtraction), 43  
iteration\_step() (in module py-  
SecDec.decomposition.iterative), 37  
iterative\_decomposition() (in module py-  
SecDec.decomposition.iterative), 37  
iterative\_sort() (in module pySecDec.matrix\_sort), 42

## L

Log (class in pySecDec.algebra), 23  
LogOfPolynomial (class in pySecDec.algebra), 23  
loop\_package() (in module pySecDec.loop\_integral), 32  
LoopIntegral (class in pySecDec.loop\_integral), 28  
LoopIntegralFromGraph (class in py-  
SecDec.loop\_integral), 29  
LoopIntegralFromPropagators (class in py-  
SecDec.loop\_integral), 30  
lowest\_order() (in module pySecDec.misc), 52

## M

make\_package() (in module pySecDec.code\_writer), 45  
missing() (in module pySecDec.misc), 52

## P

Pak\_sort() (in module pySecDec.matrix\_sort), 42  
parse\_template\_file() (in module py-  
SecDec.code\_writer.template\_parser), 48  
parse\_template\_tree() (in module py-  
SecDec.code\_writer.template\_parser), 49  
plot\_diagram() (in module py-  
SecDec.loop\_integral.draw), 34  
pole\_structure() (in module pySecDec.subtraction), 44  
Polynomial (class in pySecDec.algebra), 24  
Polytope (class in pySecDec.decomposition.geometric), 38  
Pow (class in pySecDec.algebra), 25  
powerset() (in module pySecDec.misc), 53  
primary\_decomposition() (in module py-  
SecDec.decomposition.iterative), 37  
primary\_decomposition\_polynomial() (in module py-  
SecDec.decomposition.iterative), 37  
Product (class in pySecDec.algebra), 26  
ProductRule (class in pySecDec.algebra), 27  
pySecDec.algebra (module), 21  
pySecDec.code\_writer (module), 45  
pySecDec.code\_writer.template\_parser (module), 48  
pySecDec.decomposition (module), 34  
pySecDec.decomposition.geometric (module), 38  
pySecDec.decomposition.iterative (module), 36  
pySecDec.decomposition.splitting (module), 41  
pySecDec.expansion (module), 44  
pySecDec.integral\_interface (module), 49

pySecDec.loop\_integral (module), 28  
pySecDec.matrix\_sort (module), 42  
pySecDec.misc (module), 51  
pySecDec.subtraction (module), 43

## R

rangcomb() (in module pySecDec.misc), 53  
remap\_one\_to\_zero() (in module py-  
SecDec.decomposition.splitting), 41  
remap\_parameters() (in module py-  
SecDec.decomposition.iterative), 38  
replace() (pySecDec.algebra.Function method), 22  
replace() (pySecDec.algebra.Log method), 23  
replace() (pySecDec.algebra.Polynomial method), 25  
replace() (pySecDec.algebra.Pow method), 26  
replace() (pySecDec.algebra.Product method), 26  
replace() (pySecDec.algebra.ProductRule method), 27  
replace() (pySecDec.algebra.Sum method), 28

## S

Sector (class in pySecDec.decomposition), 34  
simplify() (pySecDec.algebra.ExponentiatedPolynomial  
method), 21  
simplify() (pySecDec.algebra.Function method), 23  
simplify() (pySecDec.algebra.Log method), 23  
simplify() (pySecDec.algebra.LogOfPolynomial  
method), 24  
simplify() (pySecDec.algebra.Polynomial method), 25  
simplify() (pySecDec.algebra.Pow method), 26  
simplify() (pySecDec.algebra.Product method), 27  
simplify() (pySecDec.algebra.ProductRule method), 27  
simplify() (pySecDec.algebra.Sum method), 28  
split() (in module pySecDec.decomposition.splitting), 42  
split\_singular() (in module py-  
SecDec.decomposition.splitting), 42  
squash\_symmetry\_redundant\_sectors\_dreadnaut() (in  
module pySecDec.decomposition), 36  
squash\_symmetry\_redundant\_sectors\_sort() (in module  
pySecDec.decomposition), 35  
Suave (class in pySecDec.integral\_interface), 50  
Sum (class in pySecDec.algebra), 27  
sympify\_symbols() (in module pySecDec.misc), 53

## T

to\_sum() (pySecDec.algebra.ProductRule method), 27  
transform\_variables() (in module py-  
SecDec.decomposition.geometric), 40  
triangulate() (in module py-  
SecDec.decomposition.geometric), 40

## V

Vegas (class in pySecDec.integral\_interface), 51

vertex\_incidence\_lists() (py-  
SecDec.decomposition.geometric.Polytope  
method), 39